

# On the Grothendieck conjecture on $p$ -curvatures for $q$ -difference equations \*

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## Abstract

In the present paper, we give a  $q$ -analogue of Grothendieck conjecture on  $p$ -curvatures for  $q$ -difference equations defined over the field of rational function  $K(x)$ , where  $K$  is a finite extension of a field of rational functions  $k(q)$ , with  $k$  perfect. Then we consider the generic (also called intrinsic) Galois group in the sense of [Kat82] and [DV02]. The result in the first part of the paper lead to a description of the generic Galois group through the properties of the functional equations obtained specializing  $q$  on roots of unity. Although no general Galois correspondence holds in this setting, in the case of positive characteristic, where nonreduced groups appear, we can prove some devissage of the generic Galois group.

In the last part of the paper, we give a complete answer to the analogue of Grothendieck conjecture on  $p$ -curvatures for  $q$ -difference equations defined over the field of rational function  $K(x)$ , where  $K$  is any finitely generated extension of  $\mathbb{Q}$  and  $q \neq 0, 1$ : we prove that the generic Galois group of a  $q$ -difference module over  $K(x)$  always admits an adelic description in the spirit of the Grothendieck-Katz conjecture. To this purpose, if  $q$  is an algebraic number, we prove a generalization of the results in [DV02].

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## Introduction

In the present paper, we give a complete answer to the Grothendieck-Katz’s conjecture for  $q$ -difference equation proving the following two open cases. First of all, we allow  $q$  to be a transcendental parameter. In [DV02], there was no hope of recovering information on the classical Grothendieck conjecture for differential equations by letting  $q$  tends to 1, for lack of an appropriate topology. On the contrary, here we allow  $q$  to be a parameter and therefore we can recover a differential equation specializing  $q$  to 1. Secondly, we generalize the results in [DV02], proved for a number field, to the case of a finitely generated extension of  $\mathbb{Q}$ . Thanks to those results, we show that the Galois group of a linear  $q$ -difference equation with coefficients in a field of rational functions  $K(x)$ , with  $K$  field of characteristic zero, can always be characterized in the spirit of the Grothendieck-Katz conjecture, via some curvatures. In this way we give an “adelic” answer to the direct problem for Galois theory of  $q$ -difference equations, for which we do not have any kind of algorithm.

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The question of the algebraicity of solutions of differential or difference equations goes back at least to Schwarz, who established in 1872 an exhaustive list of hypergeometric differential equations having a full set of algebraic solutions. Galois theory of linear differential equations, and more recently Galois theory of linear difference equations, have been developed to investigate the existence of algebraic relations between the solutions of linear functional equations via the computation of a linear algebraic group called the Galois group. In particular, the existence of a basis of algebraic solutions is essentially equivalent to having a finite Galois group. The computation of these Galois groups thus provides a powerful tool to study the algebraicity of special functions. The direct problem in differential Galois theory (*i.e.* for differential equations) was solved by Hrushovski in [Hru02]. Although he actually has a computational algorithm, the calculations of the Galois group of a differential equation is still a very difficult problem, most of the time, out of reach. For difference Galois theory, the existence of a general computational algorithm is still an open question.

Grothendieck-Katz conjecture on  $p$ -curvatures conjugates these two aspects of the theory: determining whether a differential equation has a full basis of algebraic solutions and solving the direct problem. Thanks to Grothendieck's conjecture on  $p$ -curvatures we have a (necessary and) sufficient conjectural condition to test whether the solutions of a differential equation are algebraic or not. More precisely, one can reduce a differential equation

$$\mathcal{L}y = a_\mu(x) \frac{d^\mu y}{dx^\mu} + a_{\mu-1}(x) \frac{d^{\mu-1} y}{dx^{\mu-1}} + \cdots + a_0(x)y = 0,$$

with coefficients in the field  $\mathbb{Q}(x)$ , modulo  $p$  for almost all primes  $p \in \mathbb{Z}$ . Then Grothendieck's conjecture, which remains open in full generality (*cf.* [And04]) predicts:

**Conjecture 1** (Grothendieck's conjecture on  $p$ -curvatures). *The equation  $\mathcal{L}y = 0$  has a full set of algebraic solutions if (and only if) for almost all primes  $p \in \mathbb{Z}$  the reduction modulo  $p$  of  $\mathcal{L}y = 0$  has a full set of solutions in  $\mathbb{F}_p(x)$ .*

After [Kat82], this is equivalent to the fact that *the Lie algebra of the generic Galois group of a differential module  $\mathcal{M} = (M, \nabla)$  over  $\mathbb{Q}(x)$  is the smallest algebraic Lie subalgebra of  $\text{End}_{\mathbb{Q}(x)}(M)$  whose reduction modulo  $p$  contains the  $p$ -curvature for almost all  $p$ .* We are not explaining here the precise meaning of this statement, to whom a large literature is devoted.

The first and main result of the paper is the following. We consider a perfect field  $k$ , the field of rational functions  $K = k(q)$  and a linear  $q$ -difference equation

$$\mathcal{L}y(x) := a_\nu(q, x)y(q^\nu x) + \cdots + a_1(q, x)y(qx) + a_0(q, x)y(x) = 0$$

with coefficients in  $K(x)$ . Note that below we will take  $K$  to be a finite extension of  $k(q)$ , which we avoid to do here not to introduce too technical notation.

Then:

**Theorem 2.** *The equation  $\mathcal{L}y(x) = 0$  has a full set of solutions in  $K(x)$ , linearly independent over  $K$ , if and only if for almost all positive integer  $\ell$  and all primitive roots of unity  $\zeta_\ell$  of order  $\ell$  the equation*

$$a_\nu(\zeta_\ell, x)y(\zeta_\ell^\nu x) + \cdots + a_1(\zeta_\ell, x)y(\zeta_\ell x) + a_0(\zeta_\ell, x)y(x) = 0$$

*has a full set of solutions in  $k(\zeta_\ell)(x)$ , linearly independent over  $k(\zeta_\ell)$ .*

We will denote by  $\sigma_q$  the  $q$ -difference operator  $f(x) \mapsto f(qx)$ , acting on any algebra where it make sense to consider it (for instance  $K(x)$ ,  $K((x))$  etc.). A  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  over  $K(x)$  is a  $K(x)$ -vector space of finite dimension  $\nu$  equipped with a  $\sigma_q$ -semilinear bijective operator  $\Sigma_q$ :

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } m \in M \text{ and } f \in K(x).$$

One goes from a linear  $q$ -difference equation to a linear  $q$ -difference system and, then, to a linear  $q$ -difference module, and viceversa, in the same way as one goes from linear differential equations to linear differential systems and to vector bundles with connection.

One can always find a polynomial  $P(x) \in k[q, x]$  and a  $q$ -difference algebra of the form

$$\mathcal{A} = k \left[ q, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \dots \right],$$

so that there exists an  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$ , stable by  $\Sigma_q$ . We obtain in this way a  $q$ -difference module  $\mathcal{M} = (M, \Sigma_q)$  over  $\mathcal{A}$ , that allows to recover  $\mathcal{M}_{K(x)}$  by extension of scalars. We denote by  $\{\phi_v\}_v$  the family of all irreducible polynomials in  $k[q]$  whose roots are root of unity. For any  $v$  we denote by  $\kappa_v$  the order of the roots of  $\phi_v$ . We prove the following statement, which is equivalent to Theorem 2 above:

**Theorem 3** (cf. Theorem 3.1 below). *The  $q$ -difference module  $\mathcal{M}_{K(x)}$  is trivial if and only if the operator  $\Sigma_q^{\kappa_v}$  acts as the identity on  $M \otimes_{k[q]} \frac{k[q]}{(\phi_v)}$  for almost all  $v$ .*

The module  $\mathcal{M}_{K(x)}$  is said to be trivial if it is isomorphic to  $(K^\nu \otimes_K K(x), 1 \otimes \sigma_q)$ , for some positive integer  $\nu$  or, equivalently if an associated linear  $q$ -difference equation in a cyclic basis has a full set of rational solutions. Notice that  $\Sigma_q^{\kappa_v}$  induces an  $\frac{k[q]}{(\phi_v)}$ -linear map on  $M \otimes_{k[q]} \frac{k[q]}{(\phi_v)}$ .

We consider the collection  $\text{Constr}(\mathcal{M}_{K(x)})$  of  $K(x)$ -linear algebraic constructions of  $\mathcal{M}_{K(x)}$  (direct sums, tensor product, symmetric and antisymmetric product, dual). The operator  $\Sigma_q$  induces a  $q$ -difference operator on every element of  $\text{Constr}(\mathcal{M}_{K(x)})$ , that we will still call  $\Sigma_q$ . Then, the generic Galois group of  $\mathcal{M}_{K(x)}$  is defined as:

$$\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{\varphi \in \text{GL}(M_{K(x)}) : \varphi \text{ stabilizes every subset stabilized by } \Sigma_q, \text{ in any construction}\}$$

As in [Kat82], Theorem 3 is equivalent to the following statement, whose precise meaning is explained in §4:

**Theorem 4** (cf. Theorem 4.5 below). *The generic Galois group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the smallest algebraic subgroup of  $\text{GL}(M_{K(x)})$  that contains the operators  $\Sigma_q^{\kappa_v}$  modulo  $\phi_v$  for almost all  $v$ .*

In the case of positive characteristic, the group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is not necessarily reduced. Although there is no Galois correspondence for generic Galois groups, in the nonreduced case we can prove some devissage. In fact, let  $p > 0$  be the characteristic of  $k$  and let us consider the short exact sequence associated with the largest reduced subgroup  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  of the generic Galois group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ :

$$1 \longrightarrow \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \mu_{p^\ell} \longrightarrow 1.$$

Then we have (cf. Theorem 4.12 and Corollary 4.15 below):

**Theorem 5.** • *The group  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the smallest algebraic subgroup of  $\text{GL}(M_{K(x)})$  that contains the operators  $\Sigma_q^{\kappa_v p^\ell}$  modulo  $\phi_v$  for almost all  $v \in \mathcal{C}$ .*

- *$\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the generic Galois group of the  $q^{p^\ell}$ -difference module  $(M_{K(x)}, \Sigma_q^{p^\ell})$ .*
- *Let  $\tilde{K}$  be a finite extension of  $K$  containing a  $p^\ell$ -th root  $q^{1/p^\ell}$  of  $q$ . The generic Galois group  $\text{Gal}(\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}, \eta_{\tilde{K}(x^{1/p^\ell})})$  is reduced and*

$$\text{Gal}(\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}, \eta_{\tilde{K}(x^{1/p^\ell})}) \subset \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \tilde{K}(x^{1/p^\ell}).$$

In the last section of the paper we apply the previous results to the characterization of the generic Galois group of a  $q$ -difference module over  $\mathbb{C}(x)$ , with  $q \in \mathbb{C} \setminus \{0, 1\}$ . We prove a statement that can be summarized informally in the following way:

**Theorem 6.** *The generic Galois group of a complex  $q$ -difference module  $\mathcal{M} = (M, \Sigma_q)$  is the smallest algebraic subgroup of  $\text{GL}(M)$ , that contains a cofinite nonempty subset of curvatures.*

This means that there exists a field  $K$ , finitely generated over  $\mathbb{Q}$  and containing  $q$ , and a  $q$ -difference module  $\mathcal{M}_{K(x)}$  over  $K(x)$ , such that

$$\begin{cases} \mathcal{M} = \mathcal{M}_{K(x)} \otimes \mathbb{C}(x), \\ \text{Gal}(\mathcal{M}, \eta_{\mathbb{C}(x)}) = \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes \mathbb{C}(x). \end{cases}$$

Then:

- If  $q$  is a transcendental number, we are in the situation studied in the present paper.
- If  $q$  is a root of unity, the theorem is proved for algebraic generic Galois groups in [Hen96].
- If  $q$  is an algebraic number and  $K$  is a number field, we are in the situation studied in [DV02].

In §5 below, Theorem 6 is proved in the only remaining open case, namely under the assumption that  $q$  is algebraic and  $K/\mathbb{Q}$  is finitely generated, but not finite. Notice that the arguments of [DV02] use crucially the fact that number fields satisfy the product formula, therefore they cannot be applied to this situation. Here we consider a basis of transcendence of  $K/\mathbb{Q}$  and consider its elements as parameters to be specialized in the field of algebraic numbers. After the specialization we apply the results in [DV02] and then we recover the information over  $K(x)$ . To prove this last step, we use Sauloy's canonic solutions and Birkhoff connection matrix as constructed in [Sau00].

The results of this paper rise the following considerations:

*Comparison theorem with other differential Galois theories (see [DVH11b]).* There are many different Galois theories for  $q$ -difference equations. In [DVH11b], we elucidate the comparison of all of them with the generic Galois group introduced above. We point out that Theorem 6 is an important ingredient in the proof of the comparison theorems. We prove in particular that the dimension of the generic Galois group over  $K(x)$  is equal to the transcendency degree of the extension generated over the field of rational functions with elliptic coefficients over  $C^*/q^{\mathbb{Z}}$  by a full set of solutions of the  $q$ -difference equation.

*Specialization of the parameter  $q$  (see [DVH11b]).* We have proved that, when  $q$  is a parameter, *i.e.* when it is transcendental over  $k$ , independently of the characteristic, the structure of a  $q$ -difference equation is totally determined by the structure of the  $\xi$ -difference equations obtained specializing  $q$  to almost all primitive root of unity  $\xi$ . In [DVH11b], we prove that the specialization of the algebraic generic Galois group at  $q = a$  for any  $a$  in the algebraic closure of  $k$ , contains the algebraic generic Galois group of the specialized equation. In other words, we can say that the information spreads from the roots of unity to the other possible specializations of  $q$ .

*Link to the classical Grothendieck conjecture (see [DVH11b]).* If  $k$  is a number field, we can reduce the equations to positive characteristic, so that  $q$  reduces to a parameter to positive characteristic. So, if we have a  $q$ -difference equation  $Y(qx) = A(q, x)Y(x)$  with coefficients in a field  $k(q, x)$  such that  $[k : \mathbb{Q}] < \infty$ , we can either reduce it to positive characteristic and then specialize  $q$ , or specialize  $q$  and then reduce to positive characteristic. In particular, letting  $q \rightarrow 1$  in

$$\frac{Y(qx) - Y(x)}{(q-1)x} = \frac{A(q, x) - 1}{(q-1)x} Y(x)$$

we obtain a differential system. In [DVH11b], we elucidate some of the inclusions of the Galois groups that we find in this way. This could give some new method to tackle the Grothendieck conjecture for differential equations, by confluence and  $q$ -deformation of a differential equation. The idea would be to find a suitable  $q$ -deformation to translate the arithmetic of the curvatures of the linear differential equation into the  $q$ -arithmetic of the curvatures of the  $q$ -difference equation obtained by deformation.

*Link with the Galois theory of parameterized difference equations (see [DVH11a]).* In [HS08], the authors attach to a  $q$ -difference equation a parameterized Galois group, which is a differential algebraic group *à la Kolchin*. This is a subgroup of the group of invertible matrices defined by a set of algebraic differential equations. The differential dimension of this Galois group measures the hypertranscendence properties of a basis of solutions. We recall that a function  $f$  is hypertranscendental over a field  $F$  equipped with a derivation  $\partial$  if  $F[\partial^n(f), n \geq 0]/F$  is a transcendental extension of infinite degree, or equivalently, if  $f$  is not a solution of a nonlinear algebraic differential equation with coefficients in  $F$ .

In [DVH11a], we combine the Grothendieck conjecture with the differential approach to difference equations of Hardouin-Singer to obtain a parameterized generic Galois group, arithmetically characterized in the spirit of Grothendieck-Katz conjecture. This allows us to establish the relation between Malgrange-Granier  $D$ -groupoid for nonlinear  $q$ -difference equations and the Galois theory of linear  $q$ -difference equations, comparing the first one to the parameterized generic Galois group. Doing this we build the first path between Kolchins theory of linear differential algebraic groups and Malgranges  $D$ -groupoid, answering a question of B. Malgrange (see [Mal09, page 2]). It should lead to an arithmetic approach to integrability in the spirit of [CR08]. Notice that the problem of differential dependency with respect to the derivation  $\frac{d}{dq}$ , when  $q$  is transcendental, is quite peculiar and is treated in [DVH11c].

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# 1 Notation and definitions

**1.1. The base field.** Let us consider the field of rational function  $k(q)$  with coefficients in a perfect field  $k$ . We fix  $d \in ]0, 1[$  and for any irreducible polynomial  $v = v(q) \in k[q]$  we set:

$$|f(q)|_v = d^{\deg_q v(q) \cdot \text{ord}_{v(q)} f(q)}, \forall f(q) \in k[q].$$

The definition of  $|\cdot|_v$  extends to  $k(q)$  by multiplicativity. To this set of norms one has to add the  $q^{-1}$ -adic one, defined on  $k[q]$  by:

$$|f(q)|_{q^{-1}} = d^{-\deg_q f(q)}.$$

Once again, this definition extends by multiplicativity to  $k(q)$ . Then, the product formula holds:

$$\begin{aligned} \prod_{v \in k[q] \text{ irred.}} \left| \frac{f(q)}{g(q)} \right|_v &= d^{\sum_v \deg_q v(q) (\text{ord}_{v(q)} f(q) - \text{ord}_{v(q)} g(q))} \\ &= d^{\deg_q f(q) - \deg_q g(q)} \\ &= \left| \frac{f(q)}{g(q)} \right|_{q^{-1}}^{-1}. \end{aligned}$$

For any finite extension  $K$  of  $k(q)$ , we consider the family  $\mathcal{P}$  of ultrametric norms, that extends the norms defined above, up to equivalence. We suppose that the norms in  $\mathcal{P}$  are normalized so that the product formula still holds. We consider the following partition of  $\mathcal{P}$ :

- the set  $\mathcal{P}_\infty$  of places of  $K$  such that the associated norms extend, up to equivalence, either  $|\cdot|_q$  or  $|\cdot|_{q^{-1}}$ ;
- the set  $\mathcal{P}_f$  of places of  $K$  such that the associated norms extend, up to equivalence, one of the norms  $|\cdot|_v$  for an irreducible  $v = v(q) \in k[q]$ ,  $v(q) \neq q$ .<sup>1</sup>

Moreover we consider the set  $\mathcal{C}$  of places  $v \in \mathcal{P}_f$  such that  $v$  divides a valuation of  $k(q)$  having as uniformizer a factor of a cyclotomic polynomial, other than  $q - 1$ . Equivalently,  $\mathcal{C}$  is the set of places  $v \in \mathcal{P}_f$  such that  $q$  reduces to a root of unity modulo  $v$  of order strictly greater than 1. We will call  $v \in \mathcal{C}$  a cyclotomic place.

Sometimes we will write  $\mathcal{P}_K$ ,  $\mathcal{P}_{K,f}$ ,  $\mathcal{P}_{K,\infty}$  and  $\mathcal{C}_K$ , to stress out the choice of the base field.

**1.2.  $q$ -difference modules.** The field  $K(x)$  is naturally a  $q$ -difference algebra, *i.e.* is equipped with the operator

$$\begin{aligned} \sigma_q : K(x) &\longrightarrow K(x) \\ f(x) &\longmapsto f(qx). \end{aligned}$$

The field  $K(x)$  is also equipped with the  $q$ -derivation

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

satisfying a  $q$ -Leibniz formula:

$$d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x),$$

for any  $f, g \in K(x)$ .

More generally, we will consider a field  $K$ , with a fixed element  $q \neq 0$ , and an extension  $\mathcal{F}$  of  $K(x)$  equipped with a  $q$ -difference operator, still called  $\sigma_q$ , extending the action of  $\sigma_q$ , and with the skew derivation  $d_q := \frac{\sigma_q - 1}{(q-1)x}$ . Typically, in the sequel, we will consider the fields  $K(x)$ ,  $K(x^{1/r})$ ,  $r \in \mathbb{Z}_{>1}$ , or  $K((x))$ .

A  $q$ -difference module over  $\mathcal{F}$  (of rank  $\nu$ ) is a finite dimensional  $\mathcal{F}$ -vector space  $M_{\mathcal{F}}$  (of dimension  $\nu$ ) equipped with an invertible  $\sigma_q$ -semilinear operator, *i.e.*

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } f \in \mathcal{F} \text{ and } m \in M_{\mathcal{F}}.$$

A morphism of  $q$ -difference modules over  $\mathcal{F}$  is a morphism of  $\mathcal{F}$ -vector spaces, commuting with the  $q$ -difference structures (for more generalities on the topic, *cf.* [vdPS97], [DV02, Part I] or [DVRSZ03]). We denote by  $\text{Diff}(\mathcal{F}, \sigma_q)$  the category of  $q$ -difference modules over  $\mathcal{F}$ .

<sup>1</sup>The notation  $\mathcal{P}_f$ ,  $\mathcal{P}_\infty$  is only psychological, since all the norms involved here are ultrametric. Nevertheless, there exists a fundamental difference between the two sets, in fact for any  $v \in \mathcal{P}_\infty$  one has  $|q|_v \neq 1$ , while for any  $v \in \mathcal{P}_f$  the  $v$ -adic norm of  $q$  is 1. Therefore, from a  $v$ -adic analytic point of view, a  $q$ -difference equation has a totally different nature with respect to the norms in the sets  $\mathcal{P}_f$  or  $\mathcal{P}_\infty$ .

Let  $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$  be a  $q$ -difference module over  $\mathcal{F}$  of rank  $\nu$ . We fix a basis  $\underline{e}$  of  $M_{\mathcal{F}}$  over  $\mathcal{F}$  and we set:

$$\Sigma_q \underline{e} = \underline{e}A,$$

with  $A \in \mathrm{GL}_{\nu}(\mathcal{F})$ . A horizontal vector  $\vec{y} \in \mathcal{F}^{\nu}$  with respect to the basis  $\underline{e}$  for the operator  $\Sigma_q$  is a vector that verifies  $\Sigma_q(\underline{e}\vec{y}) = \underline{e}\vec{y}$ , *i.e.*  $\vec{y} = A\sigma_q(\vec{y})$ . Therefore we call

$$\sigma_q(Y) = A_1 Y, \text{ with } A_1 = A^{-1},$$

the ( $q$ -difference) system associated to  $\mathcal{M}_{\mathcal{F}}$  with respect to the basis  $\underline{e}$ . Recursively, we obtain a family of higher order  $q$ -difference systems:

$$\sigma_q^n(Y) = A_n Y \text{ and } d_q^n Y = G_n Y,$$

with  $A_n \in \mathrm{GL}_{\nu}(\mathcal{F})$  and  $G_n \in M_{\nu}(\mathcal{F})$ . Notice that:

$$A_{n+1} = \sigma_q(A_n)A_1, G_1 = \frac{A_1 - 1}{(q-1)x} \text{ and } G_{n+1} = \sigma_q(G_n)G_1(x) + d_q G_n.$$

It is convenient to set  $A_0 = G_0 = 1$ . Moreover we set  $[n]_q = \frac{q^n - 1}{q - 1}$ ,  $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$ ,  $[0]_q! = 1$  and  $G_{[n]} = \frac{G_n}{[n]_q!}$  for any  $n \geq 1$ .

**1.3. Reduction modulo places of  $K$ .** In the sequel, we will deal with an arithmetic situation, in the following sense. We consider the ring of integers  $\mathcal{O}_K$  of  $K$ , *i.e.* the integral closure of  $k[q]$  in  $K$ , and a  $q$ -difference algebra of the form

$$(1.1) \quad \mathcal{A} = \mathcal{O}_K \left[ x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \dots \right],$$

for some  $P(x) \in \mathcal{O}_K[x]$ . Then  $\mathcal{A}$  is stable by the action of  $\sigma_q$  and we can consider a free  $\mathcal{A}$ -module  $M$  equipped with a semilinear invertible operator<sup>2</sup>  $\Sigma_q$ . Notice that  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q \otimes \sigma_q)$  is a  $q$ -difference module over  $K(x)$ . We will call  $\mathcal{M} = (M, \Sigma_q)$  a  $q$ -difference module over  $\mathcal{A}$ .

Any  $q$ -difference module over  $K(x)$  comes from a  $q$ -difference module over  $\mathcal{A}$ , for a convenient choice of  $\mathcal{A}$ . The reason for considering  $q$ -difference modules over  $\mathcal{A}$  rather than over  $K(x)$ , is that we want to reduce our  $q$ -difference modules with respect to the places of  $K$ , and, in particular, with respect to the cyclotomic places of  $K$ .

We denote by  $k_v$  the residue field of  $K$  with respect to a place  $v \in \mathcal{P}$ ,  $\pi_v$  the uniformizer of  $v$  and  $q_v$  the image of  $q$  in  $k_v$ , which is defined for all places  $v \in \mathcal{P}$ . For almost all  $v \in \mathcal{P}_f$  we can consider the  $k_v(x)$ -vector space  $M_{k_v(x)} = M \otimes_{\mathcal{A}} k_v(x)$ , with the structure induced by  $\Sigma_q$ . In this way, for almost all  $v \in \mathcal{P}$ , we obtain a  $q_v$ -difference module  $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$  over  $k_v(x)$ ,

In particular, for almost all  $v \in \mathcal{C}$ , we obtain a  $q_v$ -difference module  $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$  over  $k_v(x)$ , having the particularity that  $q_v$  is a root of unity, say of order  $\kappa_v$ . This means that  $\sigma_{q_v}^{\kappa_v} = 1$  and that  $\Sigma_{q_v}^{\kappa_v}$  is a  $k_v(x)$ -linear operator. The results in [DV02, §2] apply to this situation. We recall some of them. Since we have:

$$\sigma_{q_v}^{\kappa_v} = 1 + (q-1)^{\kappa_v} x^{\kappa_v} d_{q_v}^{\kappa_v} \quad \text{and} \quad \Sigma_{q_v}^{\kappa_v} = 1 + (q-1)^{\kappa_v} x^{\kappa_v} \Delta_{q_v}^{\kappa_v},$$

where  $\Delta_{q_v} = \frac{\Sigma_{q_v} - 1}{(q_v - 1)x}$ , the following facts are equivalent:

1.  $\Sigma_{q_v}^{\kappa_v}$  is the identity;
2.  $\Delta_{q_v}^{\kappa_v}$  is zero;
3. the reduction of  $A_{\kappa_v}$  modulo  $\pi_v$  is the identity matrix;
4. the reduction of  $G_{\kappa_v}$  modulo  $\pi_v$  is zero.

**Definition 1.4.** If the conditions above are satisfied we say that  $\mathcal{M}$  has *zero  $\kappa_v$ -curvature (modulo  $\pi_v$ )*. We say that  $\mathcal{M}$  has *nilpotent  $\kappa_v$ -curvature (modulo  $\pi_v$ )* or *has nilpotent reduction*, if  $\Delta_{q_v}^{\kappa_v}$  is a nilpotent operator or equivalently if  $\Sigma_{q_v}^{\kappa_v}$  is a unipotent operator.

<sup>2</sup>We could have asked that  $\Sigma_q$  is only injective, but then, enlarging the scalar to a  $q$ -difference algebra  $\mathcal{A}' \subset K(x)$ , of the same form as (1.1), we would have obtained an invertible operator. Since we are interested in the reduction of  $\mathcal{M}$  modulo almost all places of  $K$ , we can suppose without loss of generality that  $\Sigma_q$  is invertible.



We will use this notion in §2, while in §3 we will need the following stronger notion.

**1.5.  $\kappa_v$ -curvatures (modulo  $\phi_v$ ).** We denote by  $\phi_v$  the uniformizer of the cyclotomic place of  $k(q)$  induced by  $v \in \mathcal{C}_K$ . The ring  $\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_K / (\phi_v)$  is not reduced in general, nevertheless it has a  $q$ -difference algebra structure and the results in [DV02, §2] apply again. Therefore we set:

**Definition 1.6.** A  $q$ -difference module  $\mathcal{M}$  has *zero  $\kappa_v$ -curvature (modulo  $\phi_v$ )* if the operator  $\Sigma_q^{\kappa_v}$  induces the identity (or equivalently if the operator  $\Delta_q^{\kappa_v}$  induces the zero operator) on the module  $M \otimes_{\mathcal{A}} \mathcal{A} / (\phi_v)$ .

**Remark 1.7.** The rational function  $\phi_v$  is, up to a multiplicative constant, a factor of a cyclotomic polynomial for almost all  $v$ . It is a divisor of  $[\kappa_v]_q$  and  $|\phi_v|_v = |[\kappa_v]_q|_v = |[\kappa_v]_q|_v$ .

We recall the definition of the Gauss norm associated to an ultrametric norm  $v \in \mathcal{P}$ :

$$\text{for any } \frac{\sum a_i x^i}{\sum b_j x^j} \in K(x), \quad \left| \frac{\sum a_i x^i}{\sum b_j x^j} \right|_{v, \text{Gauss}} = \frac{\sup |a_i|_v}{\sup |b_j|_v}.$$

**Proposition 1.8.** Let  $v \in \mathcal{C}_K$ . We assume that  $|G_1(x)|_{v, \text{Gauss}} \leq 1$ . Then the following assertions are equivalent:

1. The module  $\mathcal{M} = (M, \Sigma_q)$  has zero  $\kappa_v$ -curvature modulo  $\phi_v$ .
2. For any positive integer  $n$ , we have  $|G_{[n]}|_{v, \text{Gauss}} \leq 1$ , i.e. the operator  $\frac{\Delta_q^n}{[n]_q!}$  induces a well defined operator on  $\mathcal{M}_{k_v(x)} = (M_{k_v(x)}, \Sigma_{q_v})$ .

**Remark 1.9.** Even if  $q_v$  is a root of unity, the family of operators  $\frac{d_{q_v}^n}{[n]_{q_v}!}$  acting on  $k_v(x)$  is well defined. This remark is the starting point for the theory of iterated  $q$ -difference modules constructed in [Har10, §3]. Then the second assertion of the proposition above can be rewritten as:

$\mathcal{M}_{k_v(x)}$  has a natural structure of iterated  $q_v$ -difference module.

*Proof.* The only nontrivial implication is “1  $\Rightarrow$  2” whose proof is quite similar to [DV02, Lemma 5.1.2]. The Leibniz Formula for  $d_q$  and  $\Delta_q$  implies that:

$$G_{(n+1)\kappa_v} = \sum_{i=0}^{\kappa_v} \binom{\kappa_v}{i}_q \sigma_q^{\kappa_v-i} (d_q^i (G_{n\kappa_v})) G_{\kappa_v-i},$$

where  $\binom{n}{i}_q = \frac{[n]_q!}{[i]_q! [n-i]_q!}$  for any  $n \geq i \geq 0$ . If  $\mathcal{M}$  has zero  $\kappa_v$ -curvature modulo  $\phi_v$  then  $|G_{\kappa_v}|_{v, \text{Gauss}} \leq |\phi_v|_v$ . One obtains recursively that  $|G_m|_{v, \text{Gauss}} \leq |\phi_v|_v^{\lfloor \frac{m}{\kappa_v} \rfloor}$ , where we have denoted by  $[a]$  the integral part of  $a \in \mathbb{R}$ , i.e.  $[a] = \max\{n \in \mathbb{Z} : n \leq a\}$ . Since  $|[\kappa_v]_q|_v = |\phi_v|_v$  and  $|[m]_q|_v = |\phi_v|_v^{\lfloor \frac{m}{\kappa_v} \rfloor}$ , we conclude that:

$$(1.2) \quad \left| \frac{G_m}{[m]_q!} \right|_{v, \text{Gauss}} \leq 1.$$

□

## 2 Regularity of “global” nilpotent $q$ -difference modules

In this section, we are going to prove that a  $q$ -difference module is regular singular and has integral exponents if it has nilpotent reduction for sufficiently many cyclotomic places. In this setting, and in particular if the characteristic of  $k$  is zero, speaking of global nilpotence is a little bit abusive. Nevertheless, it is the terminology used in arithmetic differential equations and we think that it is evocative of the ideas that have inspired what follows.

**Definition 2.1.** A  $q$ -difference module  $(M, \Sigma_q)$  over  $\mathcal{A}$  (or another sub- $q$ -difference algebra of  $K((x))$ ) is said to be regular singular at 0 if there exists a basis  $\underline{e}$  of  $(M \otimes_{\mathcal{A}} K((x)), \Sigma_q \otimes \sigma_q)$  over  $K((x))$  such that the action of  $\Sigma_q \otimes \sigma_q$  over  $\underline{e}$  is represented by a constant matrix  $A \in \text{GL}_\nu(K)$ .

**Remark 2.2.** It follows from the Frobenius algorithm<sup>3</sup>, that a  $q$ -difference module  $M_{K(x)}$  over  $K(x)$  is regular singular if and only if there exists a basis  $\underline{e}$  such that  $\Sigma_q \underline{e} = \underline{e}A(x)$  with  $A(x) \in \mathrm{GL}_\nu(K(x)) \cap \mathrm{GL}_n(K[[x]])$ .

The eigenvalues of  $A(0)$  are called the exponents of  $\mathcal{M}$  at zero. They are well defined modulo  $q^\mathbb{Z}$ . The  $q$ -difference module  $\mathcal{M}$  is said to be regular singular *tout court* if it is regular singular both at 0 and at  $\infty$ , i.e. after a variable change of the form  $x = 1/t$ .

In the notation of the previous section, we prove the following result, which is actually an analogue of [Kat70, §13] (cf. also [DV02] for a  $q$ -difference version over a number field):

**Theorem 2.3.**

1. If a  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  has nilpotent  $\kappa_v$ -curvature modulo  $\pi_v$  for infinitely many  $v \in \mathcal{C}$  then it is regular singular.
2. Let  $\mathcal{M}$  be a  $q$ -difference module over  $\mathcal{A}$ . If there exists an infinite set of positive primes  $\wp \subset \mathbb{Z}$  such that  $\mathcal{M}$  has nilpotent  $\kappa_v$ -curvature modulo  $\pi_v$  for all  $v \in \mathcal{C}$  such that  $\kappa_v \in \wp$ , then  $\mathcal{M}$  is regular singular and its exponents (at zero and at  $\infty$ ) are all in  $q^\mathbb{Z}$ .

**Remark 2.4.** The proof of the first part of Theorem 2.3 is inspired by [Kat70, 13.1] and therefore is quite similar to [DV02, §6]. On the other hand, the proof of the triviality of the exponents (cf. Proposition 2.7 below) has significant differences with respect to the analogous results on number fields. In fact in the differential case the proof is based on Chebotarev density theorem. In [DV02] it is a consequence on some considerations on Kummer extensions and Chebotarev density theorem, while in this setting it is a consequence of Lemma 2.9 below, which can be interpreted as a statement in rational dynamic.

The proof of Theorem 2.3 is the object of the following three subsections.

## 2.1 Regularity

We prove the first part of Theorem 2.3. It is enough to prove that 0 is a regular singular point for  $\mathcal{M}$ , the proof at  $\infty$  being completely analogous.

Let  $r \in \mathbb{N}$  be a divisor of  $\nu!$  where  $\nu$  is the dimension of  $M_{K(x)}$  over  $K(x)$  and let  $L$  be a finite extension of  $K$  containing an element  $\tilde{q}$  such that  $\tilde{q}^r = q$ . We consider the field extension  $K(x) \hookrightarrow L(t)$ ,  $x \mapsto t^r$ . The field  $L(t)$  has a natural structure of  $\tilde{q}$ -difference algebra extending the  $q$ -difference structure of  $K(x)$ .

**Lemma 2.5.** *The  $q$ -difference module  $\mathcal{M}$  is regular singular at  $x = 0$  if and only if the  $\tilde{q}$ -difference module  $\mathcal{M}_{L(t)} := (M \otimes_{\mathcal{A}} L(t), \Sigma_{\tilde{q}} := \Sigma_q \otimes \sigma_{\tilde{q}})$  over  $L(t)$  is regular singular at  $t = 0$ .*

*Proof.* It is enough to notice that if  $\underline{e}$  is a basis for  $\mathcal{M}$ , then  $\underline{e} \otimes 1$  is a basis for  $\mathcal{M}_{L(t)}$  and  $\Sigma_{\tilde{q}}(\underline{e} \otimes 1) = \Sigma_q(\underline{e}) \otimes 1$ . The other implication is a consequence of the Frobenius algorithm (cf. [vdPS97] or [Sau00]).  $\square$

The next lemma can be deduced from the formal classification of  $q$ -difference modules (cf. [Pra83, Corollary 9 and §9, 3]), [Sau04b, Theorem 3.1.7]):

**Lemma 2.6.** *There exist an extension  $L(t)/K(x)$  as above, a basis  $\underline{f}$  of the  $\tilde{q}$ -difference module  $\mathcal{M}_{L(t)}$  and an integer  $\ell$  such that  $\Sigma_{\tilde{q}} \underline{f} = \underline{f}B(t)$ , with  $B(t) \in \mathrm{GL}_\nu(L(t))$  of the following form:*

$$(2.1) \quad \begin{cases} B(t) = \frac{B_\ell}{t^\ell} + \frac{B_{\ell-1}}{t^{\ell-1}} + \dots, \text{ as an element of } \mathrm{GL}_\nu(L((t))); \\ B_\ell \text{ is a constant nonnilpotent matrix.} \end{cases}$$

*Proof of the first part of Theorem 2.3.* Let  $\mathcal{B} \subset L(t)$  be a  $\tilde{q}$ -difference algebra over the ring of integers  $\mathcal{O}_L$  of  $L$ , of the same form as (1.1), containing the entries of  $B(t)$ . Then there exists a  $\mathcal{B}$ -lattice  $\mathcal{N}$  of  $\mathcal{M}_{L(t)}$  inheriting the  $\tilde{q}$ -difference module structure from  $\mathcal{M}_{L(t)}$  and having the following properties:

1.  $\mathcal{N}$  has nilpotent reduction modulo infinitely many cyclotomic places of  $L$ ;
2. there exists a basis  $\underline{f}$  of  $\mathcal{N}$  over  $\mathcal{A}$  such that  $\Sigma_{\tilde{q}} \underline{f} = \underline{f}B(t)$  and  $B(t)$  verifies (2.1).

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<sup>3</sup>cf. [vdPS97] or [Sau00, §1.1]. The algorithm is briefly summarized also in [Sau04a, §1.2.2] and [DVRSZ03].



Iterating the operator  $\Sigma_{\tilde{q}}$  we obtain:

$$\Sigma_{\tilde{q}}^m(\underline{f}) = \underline{f}B(t)B(\tilde{q}t)\cdots B(\tilde{q}^{m-1}t) = \underline{f}\left(\frac{B_\ell^m}{\tilde{q}^{\frac{\ell m(\ell m-1)}{2}}t^{m\ell}} + h.o.t.\right).$$

We know that for infinitely many cyclotomic places  $w$  of  $L$ , the matrix  $B(t)$  verifies

$$(2.2) \quad (B(t)B(\tilde{q}t)\cdots B(\tilde{q}^{\kappa_w-1}t) - 1)^{n(w)} \equiv 0 \pmod{\pi_w},$$

where  $\pi_w$  is a uniformizer of the place  $w$ ,  $\kappa_w$  is the order  $\tilde{q}$  modulo  $\pi_w$  and  $n(w)$  is a convenient positive integer. Suppose that  $\ell \neq 0$ . Then  $B_\ell^{\kappa_w} \equiv 0$  modulo  $\pi_w$ , for infinitely many  $w$ , and hence  $B_\ell$  is a nilpotent matrix, in contradiction with Lemma 2.6. So necessarily  $\ell = 0$ .

Finally we have  $\Sigma_{\tilde{q}}(\underline{f}) = \underline{f}(B_0 + h.o.t.)$ . It follows from (2.2) that  $B_0$  is actually invertible, which implies that  $\mathcal{M}_{L(t)}$  is regular singular at 0. Lemma 2.5 allows to conclude.  $\square$

## 2.2 Triviality of the exponents

Let us prove the second part of Theorem 2.3. We have already proved that 0 is a regular singularity for  $\mathcal{M}$ . This means that there exists a basis  $\underline{e}$  of  $\mathcal{M}$  over  $K(x)$  such that  $\Sigma_q \underline{e} = \underline{e}A(x)$ , with  $A(x) \in \mathrm{GL}_\nu(K[[x]]) \cap \mathrm{GL}_\nu(K(x))$ .

The Frobenius algorithm (cf. [Sau00, §1.1.1]) implies that there exists a shearing transformation  $S \in \mathrm{GL}_\nu(K[x, 1/x])$ , such that  $S(qx)A(x)S(x)^{-1} \in \mathrm{GL}_\nu(K[[x]]) \cap \mathrm{GL}_\nu(K(x))$  and that the constant term  $A_0$  of  $S(x)^{-1}A(x)S(qx)$  has the following properties: if  $\alpha$  and  $\beta$  are eigenvalues of  $A_0$  and  $\alpha\beta^{-1} \in q^\mathbb{Z}$ , then  $\alpha = \beta$ . So choosing the basis  $\underline{e}S(x)$  instead of  $\underline{e}$ , we can assume that  $A_0 = A(0)$  has this last property.

Always following the Frobenius algorithm (cf. [Sau00, §1.1.3]), one constructs recursively the entries of a matrix  $F(x) \in \mathrm{GL}_\nu(K[[x]])$ , with  $F(0) = 1$ , such that we have  $F(x)^{-1}A(x)F(qx) = A_0$ . This means that there exists a basis  $\underline{f}$  of  $\mathcal{M}_{K((x))}$  such that  $\Sigma_q \underline{f} = \underline{f}A_0$ .

The matrix  $A_0$  can be written as the product of a semi-simple matrix  $D_0$  and a unipotent matrix  $N_0$ . Since  $\mathcal{M}$  has nilpotent reduction, we deduce from §1.3 that the reduction of  $A_{\kappa_v} = A_0^{\kappa_v}$  modulo  $\pi_v$  is the identity matrix. Then  $D_0$  verifies:

$$(2.3) \quad \text{for all } v \in \mathcal{C} \text{ such that } \kappa_v \in \wp, \text{ we have } D_0^{\kappa_v} \equiv 1 \pmod{\pi_v}.$$

Let  $\tilde{K}$  be a finite extension of  $K$  in which we can find all the eigenvalues of  $D_0$ . Then any eigenvalue  $\alpha \in \tilde{K}$  of  $A_0$  has the property that  $\alpha^{\kappa_v} = 1$  modulo  $w$ , for all  $w \in \mathcal{C}_{\tilde{K}}$ ,  $w|v$  and  $v$  satisfies (2.3). In other words, the reduction modulo  $w$  of an eigenvalue  $\alpha$  of  $A_0$  belongs to the multiplicative cyclic group generated by the reduction of  $q$  modulo  $\pi_v$ .

To end the proof, we have to prove that  $\alpha \in q^\mathbb{Z}$ . So we are reduced to prove the proposition below.

**Proposition 2.7.** *Let  $K/k(q)$  be a finite extension and  $\wp \subset \mathbb{Z}$  be an infinite set of positive primes. For any  $v \in \mathcal{C}$ , let  $\kappa_v$  be the order of  $q$  modulo  $\pi_v$ , as a root of unity.*

*If  $\alpha \in K$  is such that  $\alpha^{\kappa_v} \equiv 1$  modulo  $\pi_v$  for all  $v \in \mathcal{C}$  such that  $\kappa_v \in \wp$ , then  $\alpha \in q^\mathbb{Z}$ .*

**Remark 2.8.** Let  $K = \mathbb{Q}(\tilde{q})$ , with  $\tilde{q}^r = q$ , for some integer  $r > 1$ . If  $\tilde{q}$  is an eigenvalue of  $A_0$  we would be asking that for infinitely many positive primes  $\ell \in \mathbb{Z}$  there exists a primitive root of unity  $\zeta_{r\ell}$  of order  $r\ell$ , which is also a root of unity of order  $\ell$ . Of course, this cannot be true, unless  $r = 1$ .

## 2.3 Proof of Proposition 2.7

We denote by  $k_0$  either the field of rational numbers  $\mathbb{Q}$ , if the characteristic of  $k$  is zero, or the field with  $p$  elements  $\mathbb{F}_p$ , if the characteristic of  $k$  is  $p > 0$ . First of all, let us suppose that  $k$  is a finite perfect extension of  $k_0$  of degree  $d$  and fix an embedding  $k \hookrightarrow \bar{k}$  of  $k$  in its algebraic closure  $\bar{k}$ . In the case of a rational function  $f \in k(q)$ , Proposition 2.7 is a consequence of the following lemma:

**Lemma 2.9.** *Let  $[k : k_0] = d < \infty$  and let  $f(q) \in k(q)$  be nonzero rational function. If there exists an infinite set of positive primes  $\wp \subset \mathbb{Z}$  with the following property:*

*for any  $\ell \in \wp$  there exists a primitive root of unity  $\zeta_\ell$  of order  $\ell$  such that  $f(\zeta_\ell)$  is a root of unity of order  $\ell$ ,*

*then  $f(q) \in q^\mathbb{Z}$ .*

**Remark 2.10.** If  $k = \mathbb{C}$  and  $y - f(q)$  is irreducible in  $\mathbb{C}[q, y]$ , the result can be deduced from [Lan83, Chapter 8, Theorem 6.1], whose proof uses Bézout theorem. We give here a totally elementary proof, that holds also in positive characteristic.

Proposition 2.7 can be rewritten in the language of rational dynamic. We denote by  $\mu_\ell$  the group of root of unity of order  $\ell$ . The following assertions are equivalent:

1.  $f(q) \in k(q)$  satisfies the assumptions of Lemma 2.9.
2. There exist infinitely many  $\ell \in \mathbb{N}$  such that the group  $\mu_\ell$  of roots of unity of order  $\ell$  verifies  $f(\mu_\ell) \subset \mu_\ell$ .
3.  $f(q) \in q^{\mathbb{Z}}$ .
4. The Julia set of  $f$  is the unit circle.

As it was pointed out to us by C. Favre, the equivalence between the last two assumptions is a particular case of [Zdu97], while the equivalence between the second and the fourth assumption can be deduced from [FRL06] or [Aut01].

*Proof.* Let  $f(q) = \frac{P(q)}{Q(q)}$ , with  $P = \sum_{i=0}^D a_i q^i, Q = \sum_{i=0}^D b_i q^i \in k[q]$  coprime polynomials of degree less equal to  $D$ , and let  $\ell$  be a prime such that:

- $f(\zeta_\ell) \in \mu_\ell$ ;
- $2D < \ell - 1$ .

Moreover, since  $\wp$  is infinite, we can chose  $\ell \gg 0$  so that the extensions  $k$  and  $k_0(\mu_\ell)$  are linearly disjoint over  $k_0$ . Since  $k$  is perfect, this implies that the minimal polynomial of the primitive  $\ell$ -th root of unity  $\zeta_\ell$  over  $k$  is  $\chi(X) = 1 + X + \dots + X^{\ell-1}$ . Now let  $\kappa \in \{0, \dots, \ell - 1\}$  be such that  $f(\zeta_\ell) = \zeta_\ell^\kappa$ , i.e.

$$\sum_{i=0}^D a_i \zeta_\ell^i = \sum_{i=0}^D b_i \zeta_\ell^{i+\kappa}.$$

We consider the polynomial  $H(q) = \sum_{i=0}^D a_i q^i - \sum_{j=\kappa}^{D+\kappa} b_{j-\kappa} q^j$  and distinguish three cases:

1. If  $D + \kappa < \ell - 1$ , then  $H(q)$  has  $\zeta_\ell$  as a zero and has degree strictly inferior to  $\ell - 1$ . Necessarily  $H(q) = 0$ . Thus we have

$$a_0 = a_1 = \dots = a_{\kappa-1} = b_{D+1-\kappa} = \dots = b_D = 0 \quad \text{and} \quad a_i = b_{i-\kappa} \text{ for } i = \kappa, \dots, D,$$

which implies  $f(q) = q^\kappa$ .

2. If  $D + \kappa = \ell - 1$  then  $H(q)$  is a  $k$ -multiple of  $\chi(q)$  and therefore all the coefficients of  $H(q)$  are all equal. Notice that the inequality  $D + \kappa \geq \ell - 1$  forces  $\kappa$  to be strictly bigger than  $D$ , in fact otherwise one would have  $\kappa + D \leq 2D < \ell - 1$ . For this reason the coefficients of  $H(q)$  of the monomials  $q^{D+1}, \dots, q^\kappa$  are all equal to zero. Thus

$$a_0 = a_1 = \dots = a_D = b_0 = \dots = b_D = 0$$

and therefore  $f = 0$  against the assumptions. So the case  $D + \kappa = \ell - 1$  cannot occur.

3. If  $D + \kappa > \ell - 1$ , then  $\kappa > D > D + \kappa - \ell$ , since  $\kappa > D$  and  $\kappa - \ell < 0$ . In this case we shall rather consider the polynomial  $\tilde{H}(q)$  defined by:

$$\tilde{H}(q) = \sum_{i=0}^D a_i q^i - \sum_{i=\kappa}^{\ell-1} b_{i-\kappa} q^i - \sum_{i=0}^{D+\kappa-\ell} b_{i+\ell-\kappa} q^i.$$

Notice that  $H(\zeta_\ell) = \tilde{H}(\zeta_\ell) = 0$  and that  $\tilde{H}(q)$  has degree smaller or equal than  $\ell - 1$ . As in the previous case,  $\tilde{H}(q)$  is a  $k$ -multiple of  $\chi(q)$ . We get

$$b_j = 0 \text{ for } j = 0, \dots, \ell - 1 - \kappa$$

and

$$a_0 - b_{\ell-\kappa} = \dots = a_{D+\kappa-\ell} - b_D = a_{D+\kappa-\ell+1} = \dots = a_D = 0.$$

We conclude that  $f(q) = q^{\kappa-\ell}$ .

This ends the proof.  $\square$

We are going to deduce Proposition 2.7 from Lemma 2.9 in two steps: first of all we are going to show that we can drop the assumption that  $[k : k_0]$  is finite and then that one can always reduce to the case of a rational function.

**Lemma 2.11.** *Lemma 2.9 holds if  $k/k_0$  is a finitely generated (not necessarily algebraic) extension.*

**Remark 2.12.** Since  $f(q) \in k(q)$ , replacing  $k$  by the field generated by the coefficients of  $f$  over  $k_0$ , we can always assume that  $k/k_0$  is finitely generated.

*Proof.* Let  $\tilde{k}$  be the algebraic closure of  $k_0$  in  $k$  and let  $k'$  be an intermediate field of  $k/\tilde{k}$ , such that  $f(q) \in k'(q) \subset k(q)$  and that  $k'/\tilde{k}$  has minimal transcendence degree  $\iota$ . We suppose that  $\iota > 0$ , to avoid the situation of Lemma 2.9. So let  $a_1, \dots, a_\iota$  be transcendence basis of  $k'/\tilde{k}$  and let  $k'' = \tilde{k}(a_1, \dots, a_\iota)$ . If  $k'/\tilde{k}$  is purely transcendental, *i.e.* if  $k' = k''$ , then  $f(q) = P(q)/Q(q)$ , where  $P(q)$  and  $Q(q)$  can be written in the form:

$$P(q) = \sum_i \sum_{\underline{j}} \alpha_{\underline{j}}^{(i)} a_{\underline{j}} q^i \quad \text{and} \quad Q(q) = \sum_i \sum_{\underline{j}} \beta_{\underline{j}}^{(i)} a_{\underline{j}} q^i,$$

with  $\underline{j} = (j_1, \dots, j_\iota) \in \mathbb{Z}_{\geq 0}^\iota$ ,  $a_{\underline{j}} = a_{j_1} \cdots a_{j_\iota}$  and  $\alpha_{\underline{j}}^{(i)}, \beta_{\underline{j}}^{(i)} \in \tilde{k}$ . If we reorganize the terms of  $P$  and  $Q$  so that

$$P(q) = \sum_{\underline{j}} a_{\underline{j}} D_{\underline{j}}(q) \quad \text{and} \quad Q(q) = \sum_{\underline{j}} a_{\underline{j}} C_{\underline{j}}(q),$$

we conclude that the assumption  $f(\zeta_\ell) \in \mu_\ell$  for infinitely many primes  $\ell$  implies that  $f_{\underline{j}} = \frac{D_{\underline{j}}}{C_{\underline{j}}}$  is a rational function with coefficients in  $\tilde{k}$  satisfying the assumptions of Lemma 2.9. Moreover, since the  $f_{\underline{j}}$ 's take the same values at infinitely many roots of unity, they are all equal. Finally, we conclude that  $f_{\underline{j}}(q) = q^d$  for any  $\underline{j}$  and hence that  $f = q^d \frac{\sum_{\underline{j}} \alpha_{\underline{j}}}{\sum_{\underline{j}} \beta_{\underline{j}}} = q^d$ .

Now let us suppose that  $k' = k''(b)$  for some primitive element  $b$ , algebraic over  $k''$ , of degree  $e$ . Then once again we write  $f(q) = P(q)/Q(q)$ , with:

$$P(q) = \sum_i \sum_{h=0}^{e-1} \alpha_{i,h} b^h q^i \quad \text{and} \quad Q(q) = \sum_i \sum_{h=0}^{e-1} \beta_{i,h} b^h q^i,$$

with  $\alpha_{i,h}, \beta_{i,h} \in k''$ . Again we conclude that  $\frac{\sum_i \alpha_{i,h} q^i}{\sum_i \beta_{i,h} q^i} = q^d$  for any  $h = 0, \dots, e-1$ , and hence that  $f(q) = q^d$ .  $\square$

*End of the proof of Proposition 2.7.* Let  $\tilde{K} = k(q, f) \subset K$ . If the characteristic of  $k$  is  $p$ , replacing  $f$  by a  $p^n$ -th power of  $f$ , we can suppose that  $\tilde{K}/k(q)$  is a Galois extension. So we set:

$$y = \prod_{\varphi \in \text{Gal}(\tilde{K}/k(q))} f^\varphi \in k(q).$$

For infinitely many  $v \in \mathcal{C}_{k(q)}$  such that  $\kappa_v$  is a prime, we have  $f^{\kappa_v} \equiv 1$  modulo  $w$ , for any  $w|v$ . Since  $\text{Gal}(\tilde{K}/K)$  acts transitively over the set of places  $w \in \mathcal{C}_{\tilde{K}}$  such that  $w|v$ , this implies that  $y^{\kappa_v} \equiv 1$  modulo  $\pi_v$ . Then Lemmas 2.11 and 2.9 allow us to conclude that  $y \in q^{\mathbb{Z}}$ . This proves that we are in the following situation:  $f$  is an algebraic function such that  $|f|_w = 1$  for any  $w \in \mathcal{P}_{\tilde{K},f}$  and that  $|f|_w \neq 1$  for any  $w \in \mathcal{P}_{\tilde{K},\infty}$ . We conclude that  $f = cq^{s/r}$  for some nonzero integers  $s, r$  and some constant  $c$  in a finite extension of  $k$ . Since  $f^{\kappa_v} \equiv 1$  modulo  $w$  for all  $w \in \mathcal{C}_{\tilde{K}}$  such that  $\kappa_v \in \wp$ , we finally obtain that  $r = 1$  and  $c = 1$ .  $\square$

### 3 Main result

In this section we are proving an analogue of the Grothendieck conjecture on  $p$ -curvatures under the assumption that  $q$  is transcendental. Roughly speaking, we are going to prove that a  $q$ -difference module is trivial if and only if its reduction modulo almost all cyclotomic places is trivial.

We say that the  $q$ -difference module  $\mathcal{M} = (M, \Sigma_q)$  of rank  $\nu$  over a  $q$ -difference field  $\mathcal{F}$  is trivial if there exists a basis  $\underline{f}$  of  $M$  over  $\mathcal{F}$  such that  $\Sigma_q \underline{f} = \underline{f}$ . This is equivalent to ask that the  $q$ -difference system associated to  $\mathcal{M}$  with respect to a basis (and hence any basis)  $\underline{e}$  has a fundamental solution in  $\mathrm{GL}_\nu(\mathcal{F})$ . We say that a  $q$ -difference module  $\mathcal{M} = (M, \Sigma_q)$  over  $\mathcal{A}$  becomes trivial over a  $q$ -difference field  $\mathcal{F}$  over  $\mathcal{A}$  if the  $q$ -difference module  $(M \otimes_{\mathcal{A}} \mathcal{F}, \Sigma_q \otimes \sigma_q)$  is trivial.

**Theorem 3.1.** *A  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  has zero  $\kappa_v$ -curvature modulo  $\phi_v$  for almost all  $v \in \mathcal{C}$  if and only if  $\mathcal{M}$  becomes trivial over  $K(x)$ .*

**Remark 3.2.** As proved in [DV02, Proposition 2.1.2], if  $\Sigma_q^{\kappa_v}$  is the identity modulo  $\phi_v$  then the  $q_v$ -difference module  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$  is trivial.

Theorem 3.1 is equivalent to the following statement, which is a  $q$ -analogue of the conjecture stated at the very end of [MvdP03]:

**Corollary 3.3.** *For a  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  the following statement are equivalent:*

1. *The  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  becomes trivial over  $K(x)$ ;*
2. *It induces an iterative  $q_v$ -difference structure over  $\mathcal{M}_{k_v(x)}$  for almost all  $v \in \mathcal{C}$ ;*
3. *It induces a trivial iterative  $q_v$ -difference structure over  $\mathcal{M}_{k_v(x)}$  for almost all  $v \in \mathcal{C}$ .*

**Remark 3.4.** The first assertion is equivalent to the fact that the Galois group of  $\mathcal{M}_{K(x)}$  is trivial, while the fourth assertion is equivalent to the fact that the iterative Galois group of  $\mathcal{M}_{k_v(x)}$  over  $k_v(x)$  is 1 for almost all  $v \in \mathcal{C}$ .

*Proof.* The equivalence  $1 \Leftrightarrow 2$  is a consequence of Proposition 1.8 and Theorem 3.1, while the implication  $3 \Rightarrow 2$  is tautological.

Let us prove that  $1 \Rightarrow 3$ . If the  $q$ -difference module  $\mathcal{M}$  becomes trivial over  $K(x)$ , then there exist an  $\mathcal{A}$ -algebra  $\mathcal{A}'$ , of the form (1.1), obtained from  $\mathcal{A}$  inverting a polynomial and its  $q$ -iterates, and a basis  $\underline{e}$  of  $M \otimes_{\mathcal{A}} \mathcal{A}'$  over  $\mathcal{A}'$ , such that the associated  $q$ -difference system is  $\sigma_q(Y) = Y$ . Therefore, for almost all  $v \in \mathcal{C}$ ,  $\mathcal{M}$  induces an iterative  $q_v$ -difference module  $\mathcal{M}_{k_v(x)}$  whose iterative  $q_v$ -difference equations are given by  $\frac{d_{q_v}^{\kappa_v}}{[\kappa_v]_{q_v}}(Y) = 0$  for all  $n \in \mathbb{N}$  (cf. [Har10, Proposition 3.17]).  $\square$

As far as the proof of Theorem 3.1 is regarded, one implication is trivial. The proof of the other is divided into steps. So let us suppose that the  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  has zero  $\kappa_v$ -curvature modulo  $\phi_v$  for almost all  $v \in \mathcal{C}$ , then:

*Step 1.* The  $q$ -difference module  $\mathcal{M}$  becomes trivial over  $K((x))$ , meaning that the module  $\mathcal{M}_{K((x))} = (M \otimes_{\mathcal{A}} K((x)), \Sigma_q \otimes \sigma_q)$  is trivial (cf. Corollary 3.6 below).

*Step 2.* There exists a basis  $\underline{e}$  of  $\mathcal{M}_{K(x)}$ , such that the associated  $q$ -difference system has a fundamental matrix of solution  $Y(x)$  in  $\mathrm{GL}(K[[x]])$  whose entries are Taylor expansions of rational functions (cf. Proposition 3.7 below).

**Remark 3.5.** Theorem 3.1 is a function field analogue of the main result of [DV02]. Step 1 is inspired by [Kat70, 13.1] (cf. also [DV02, §6] for  $q$ -difference equations over number fields). The main difference is Proposition 2.7 proved above. Step 2 is closed to [DV02, §8] and uses the Borel-Dwork criteria (cf. [And89, VIII, 1.2]).

### Step 1: triviality over $K((x))$

The triviality over  $K((x))$  is a consequence of Theorem 2.3:

**Corollary 3.6.** *If there exists an infinite set of positive primes  $\wp \subset \mathbb{Z}$  such that the  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  has zero  $\kappa_v$ -curvature modulo  $\pi_v$  (and a fortiori modulo  $\phi_v$ ) for all  $v \in \mathcal{C}$  with  $\kappa_v \in \wp$ , then  $\mathcal{M}$  becomes trivial over the field of formal Laurent series  $K((x))$ .*

*Proof.* If  $\mathcal{M}$  has zero  $\kappa_v$ -curvature modulo  $\pi_v$  then (cf. (2.3) for notation) we actually have:

$$\text{for all } v \in \mathcal{C} \text{ such that } \kappa_v \in \wp, D_0^{\kappa_v} \equiv 1 \text{ and } N_0^{\kappa_v} \equiv 1 \text{ modulo } \pi_v,$$

where  $\Sigma_q \underline{e} = \underline{e} A_0$ , for a chosen basis  $\underline{e}$  of  $\mathcal{M}_{K((x))}$  and a constant matrix  $A_0 = D_0 N_0 \in \mathrm{GL}_\nu(K)$ . This immediately implies, because of Proposition 2.7, that all the exponents are in  $q^{\mathbb{Z}} \subset k(q) \subset K$  and that the matrix  $A_0$  of  $\mathcal{M}$ , w.r.t. the  $K(x)$ -basis  $\underline{e}$ , is diagonalisable. Therefore there exist a diagonal matrix  $D$  with coefficients in  $\mathbb{Z}$  and a matrix  $C \in \mathrm{GL}_\nu(K)$  such that the basis  $\underline{e}' = \underline{e} C x^D$  of  $\mathcal{M}_{K((x))}$  is invariant under the action of  $\Sigma_q$ .  $\square$

## Step 2: rationality of solutions

**Proposition 3.7.** *If a  $q$ -difference module  $\mathcal{M}$  over  $\mathcal{A}$  has zero  $\kappa_v$ -curvature modulo  $\phi_v$  for almost all  $v \in \mathcal{C}$  then there exists a basis  $\underline{e}$  of  $M_{K(x)}$  over  $K(x)$  such that the associated  $q$ -difference system has a formal fundamental solution  $Y(x) \in \mathrm{GL}_\nu(K((x)))$ , which is the Taylor expansion at 0 of a matrix in  $\mathrm{GL}_\nu(K(x))$ , i.e.  $\mathcal{M}$  becomes trivial over  $K(x)$ .*

**Remark 3.8.** This is the only part of the proof of Theorem 3.1 where we need to suppose that the  $\kappa_v$ -curvature are zero modulo  $\phi_v$  for almost all  $v$ .

*Proof.* (cf. [DV02, Proposition 8.2.1]) Let  $\underline{e}$  be a basis of  $M$  over  $K(x)$ . Because of Corollary 3.6, applying a basis change with coefficients in  $K[x, \frac{1}{x}]$ , we can actually suppose that  $\Sigma_q \underline{e} = \underline{e} A(x)$ , where  $A(x) \in \mathrm{GL}_\nu(K(x))$  has no pole at 0 and  $A(0)$  is the identity matrix. In the notation of §1.2, the recursive relation defining the matrices  $G_n(x)$  implies that they have no pole at 0. This means that  $Y(x) := \sum_{n \geq 0} G_n(0)x^n$  is a fundamental solution of the  $q$ -difference system associated to  $\mathcal{M}_{K(x)}$  with respect to the basis  $\underline{e}$ , whose entries verify the following properties:

- For any  $v \in \mathcal{P}_\infty$ , the matrix  $Y(x)$  has infinite  $v$ -adic radius of meromorphy. This assertion is a general fact about regular singular  $q$ -difference systems with  $|q|_v \neq 1$ . The proof is based on the estimate of the growth of the  $q$ -factorials compared to the growth of  $G_n(0)$ , which gives the analyticity at 0, and on the fact that the  $q$ -difference system itself gives a meromorphic continuation of the solution.
- Since  $|[n]_q|_{v, \text{Gauss}} = 1$  for any noncyclotomic place  $v \in \mathcal{P}_f$ , we have  $|G_{[m]}(x)|_{v, \text{Gauss}} \leq 1$  for almost all  $v \in \mathcal{P}_f \setminus \mathcal{C}$ . For the finitely many  $v \in \mathcal{P}_f$  such that  $|G_1(x)|_{v, \text{Gauss}} > 1$ , there exists a constant  $C > 0$  such that  $|G_{[m]}(x)|_{v, \text{Gauss}} \leq C^m$ , for any positive integer  $m$ .
- For almost all  $v \in \mathcal{C}$  and all positive integer  $m$ ,  $|G_{[m]}(x)|_{v, \text{Gauss}} \leq 1$  (cf. Proposition 1.8), while for the remaining finitely many  $v \in \mathcal{C}$  there exists a constant  $C > 0$  such that  $|G_{[m]}(x)|_{v, \text{Gauss}} \leq C^m$  for any positive integer  $m$ .

This implies that:

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{v \in \mathcal{P}} \log^+ |G_{[m]}(x)|_{v, \text{Gauss}} = \limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{v \in \mathcal{C}} \log^+ |G_{[m]}(x)|_{v, \text{Gauss}} < \infty.$$

To conclude that  $Y(x)$  is the expansion at zero of a matrix with rational entries we apply a simplified form of the Borel-Dwork criteria for function fields, which says exactly that a formal power series having positive radius of convergence for almost all places and infinite radius of meromorphy at one fixed place is the expansion of a rational function. The proof in this case is a slight simplification of [DV02, Proposition 8.4.1]<sup>4</sup>, which is itself a simplification of the more general criteria [And04, Theorem 5.4.3]. We are omitting the details.  $\square$

## 4 Generic Galois group

Let  $\mathcal{M} = (M, \Sigma_q)$  be a  $q$ -difference module of rank  $\nu$  over  $\mathcal{A}$ , as in the previous sections. Since  $M_{K(x)} = (M_{K(x)}, \Sigma_q)$  is a  $q$ -difference module over  $K(x)$ , we can consider the collection  $\text{Constr}_{K(x)}(\mathcal{M}_{K(x)})$  of all  $q$ -difference modules obtained from  $\mathcal{M}_{K(x)}$  by algebraic construction. This means that we consider the family of  $q$ -difference modules containing  $\mathcal{M}_{K(x)}$  and closed under direct sum, tensor product, dual, symmetric and antisymmetric products. For the reader convenience, we remind the definition of the duality and the tensor product, from which we can deduce all the other algebraic constructions:

- The  $q$ -difference structure on the dual  $M_{K(x)}^*$  of  $M_{K(x)}$  is defined by:

$$\langle \Sigma_q^*(m^*), m \rangle = \sigma_q(\langle m^*, \Sigma_q^{-1}(m) \rangle),$$

for any  $m^* \in M_{K(x)}^*$  and any  $m \in M_{K(x)}$ .

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<sup>4</sup>The simplification comes from the fact that there are no archimedean norms in this setting.

- If  $\mathcal{N}_{K(x)} = (N_{K(x)}, \Sigma_q)$ , the  $q$ -difference structure on the tensor product  $M_{K(x)} \otimes_{K(x)} N_{K(x)}$  is defined by

$$\Sigma_q(m \otimes n) = \Sigma_q(m) \otimes \Sigma_q(n),$$

for any  $m \in M_{K(x)}$  and any  $n \in N_{K(x)}$  (cf. for instance [DV02, §9.1] or [Sau04b, §2.1.6]).

We will denote  $\text{Constr}_{K(x)}(M_{K(x)})$  the collection of algebraic constructions of the  $K(x)$ -vector space  $M_{K(x)}$ , i.e. the collection of underlying vector spaces of the family  $\text{Constr}_{K(x)}(\mathcal{M}_{K(x)})$ . Notice that  $\text{GL}(M_{K(x)})$  acts naturally, by functoriality, on any element of  $\text{Constr}_{K(x)}(M_{K(x)})$ .

**Definition 4.1.** The *generic Galois group*<sup>5</sup>  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  of  $\mathcal{M}_{K(x)}$  is the subgroup of  $\text{GL}(M_{K(x)})$  which is the stabiliser of all the  $q$ -difference submodules over  $K(x)$  of any object in  $\text{Constr}_{K(x)}(\mathcal{M}_{K(x)})$ .

The group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is a tannakian object. In fact, the full tensor category  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$  generated by  $\mathcal{M}_{K(x)}$  in  $\text{Diff}(K(x), \sigma_q)$  is naturally a tannakian category, when equipped with the forgetful functor

$$\eta : \langle \mathcal{M}_{K(x)} \rangle^\otimes \longrightarrow \{K(x)\text{-vector spaces}\}.$$

The functor  $\text{Aut}^\otimes(\eta)$  defined over the category of  $K(x)$ -algebras is representable by the algebraic group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ .

Notice that in positive characteristic  $p$ , the group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is not necessarily reduced. An easy example is given by the equation  $y(qx) = q^{1/p}y(x)$ , whose generic Galois group is  $\mu_p$  (cf. [vdPR07, §7]).

**Remark 4.2.** We recall that the Chevalley theorem, that also holds for nonreduced groups (cf. [DG70, II, §2, n.3, Corollary 3.5]), ensures that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  can be defined as the stabilizer of a rank one submodule (which is not necessarily a  $q$ -difference module) of a  $q$ -difference module contained in an algebraic construction of  $\mathcal{M}_{K(x)}$ . Nevertheless, it is possible to find a line that defines  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  as the stabilizer and that is also a  $q$ -difference module. In fact the noetherianity of  $\text{GL}(M_{K(x)})$  implies that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is defined as the stabilizer of a finite family of  $q$ -difference submodules  $\mathcal{W}_{K(x)}^{(i)} = (W_{K(x)}^{(i)}, \Sigma_q)$  contained in some objects  $\mathcal{M}_{K(x)}^{(i)}$  of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ . It follows that the line

$$L_{K(x)} = \bigwedge^{\dim \oplus_i W_{K(x)}^{(i)}} \left( \bigoplus_i W_{K(x)}^{(i)} \right) \subset \bigwedge^{\dim \oplus_i W_{K(x)}^{(i)}} \left( \bigoplus_i M_{K(x)}^{(i)} \right)$$

is a  $q$ -difference module and defines  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  as a stabilizer (cf. [Kat82, proof of Proposition 9]).

In the sequel, we will use the notation  $\text{Stab}(W_{K(x)}^{(i)}, i)$  to say that a group is the stabilizer of the set of vector spaces  $\{W_{K(x)}^{(i)}\}_i$ .

Let  $G$  be a closed algebraic subgroup of  $\text{GL}(M_{K(x)})$  such that  $G = \text{Stab}(L_{K(x)})$ , for some line  $L_{K(x)}$  contained in an object  $\mathcal{W}_{K(x)}$  of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ . The  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$  determines an  $\mathcal{A}$ -lattice  $L$  of  $L_{K(x)}$  and an  $\mathcal{A}$ -lattice  $W$  of  $W_{K(x)}$ . The latter is the underlying space of a  $q$ -difference module  $\mathcal{W} = (W, \Sigma_q)$  over  $\mathcal{A}$ .

**Definition 4.3.** Let  $\tilde{\mathcal{C}}$  be a cofinite subset of  $\mathcal{C}_K$  and  $(\Lambda_v)_{v \in \tilde{\mathcal{C}}}$  be a family of  $\mathcal{A}/(\phi_v)$ -linear operators acting on  $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ . We say that the algebraic group  $G \subset \text{GL}(M_{K(x)})$  contains the operators  $\Lambda_v$  modulo  $\phi_v$  for almost all  $v \in \mathcal{C}_K$  if for almost all  $v \in \tilde{\mathcal{C}}$  the operator  $\Lambda_v$  stabilizes  $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$  inside  $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ :

$$\Lambda_v \in \text{Stab}_{\mathcal{A}/(\phi_v)}(L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)).$$

**Remark 4.4.** As in [DV02, 10.1.2], one can prove that the definition above is independent of the choice of  $\mathcal{A}$ ,  $M$  and  $L_{K(x)}$ .

The main result of this section is the following:

**Theorem 4.5.** The algebraic group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the smallest closed algebraic subgroup of  $\text{GL}(M_{K(x)})$  that contains the operators  $\Sigma_q^{\kappa_v}$  modulo  $\phi_v$ , for almost all  $v \in \mathcal{C}$ .

**Remark 4.6.** The noetherianity of  $\text{GL}(M_{K(x)})$  implies that the smallest closed algebraic subgroup of  $\text{GL}(M_{K(x)})$  that contains the operators  $\Sigma_q^{\kappa_v}$  modulo  $\phi_v$ , for almost all  $v \in \mathcal{C}$ , is well-defined.

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<sup>5</sup>In [And01] it is called the *intrinsic* Galois group of  $\mathcal{M}_{K(x)}$ .



A part of Theorem 4.5 is easy to prove:

**Lemma 4.7.** *The algebraic group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  contains the operators  $\Sigma_q^{\kappa_v}$  modulo  $\phi_v$  for almost all  $v \in \mathcal{C}_K$ .*

*Proof.* The statement follows immediately from the fact that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  can be defined as the stabilizer of a rank one  $q$ -difference module in  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ , which is *a fortiori* stable by the action of  $\Sigma_q^{\kappa_v}$ .  $\square$

**Corollary 4.8.**  *$\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$  if and only if  $\mathcal{M}_{K(x)}$  is a trivial  $q$ -difference module.*

*Proof.* Because of the lemma above, if  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \{1\}$  is the trivial group, then  $\Sigma_q^{\kappa_v}$  induces the identity on  $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ . Therefore Theorem 3.1 implies that  $\mathcal{M}_{K(x)}$  is trivial. On the other hand, if  $\mathcal{M}_{K(x)}$  is trivial, then it is isomorphic to the  $q$ -difference module  $(K^\nu \otimes_K K(x), 1 \otimes \sigma_q)$ . It follows that the generic Galois group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is forced to stabilize all the lines generated by vectors of the type  $v \otimes 1$ , with  $v \in K^\nu$ . Therefore it is the trivial group.  $\square$

Now we are ready to give the proof of Theorem 4.5, whose main ingredient is Theorem 3.1. The argument is inspired by [Kat82, §X].

*Proof of Theorem 4.5.* Lemma 4.7 says that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  contains the smallest subgroup  $G$  of  $\text{GL}(M_{K(x)})$  that contains the operator  $\Sigma_q^{\kappa_v}$  modulo  $\phi_v$  for almost all  $v \in \mathcal{C}_K$ . Let  $L_{K(x)}$  be a line contained in some object of the category  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ , that defines  $G$  as a stabilizer. Then there exists a smaller  $q$ -difference module  $\mathcal{W}_{K(x)}$  over  $K(x)$  that contains  $L_{K(x)}$ . Let  $L$  and  $\mathcal{W} = (W, \Sigma_q)$  be the associated  $\mathcal{A}$ -modules. Any generator  $m$  of  $L$  as an  $\mathcal{A}$ -module is a cyclic vector for  $\mathcal{W}$  and the operator  $\Sigma_q^{\kappa_v}$  acts on  $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$  with respect to the basis induced by the cyclic basis generated by  $m$  via a diagonal matrix. Because of the definition of the  $q$ -difference structure on the dual module  $\mathcal{W}^*$  of  $\mathcal{W}$ , the group  $G$  can be define as the subgroup of  $\text{GL}(M_{K(x)})$  that fixes a line  $L'$  in  $W^* \otimes W$ , i.e. such that  $\Sigma_q^{\kappa_v}$  acts as the identity on  $L' \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ , for almost all cyclotomic places  $v$ . It follows from Theorem 3.1 that the minimal submodule  $\mathcal{W}'$  that contains  $L'$  becomes trivial over  $K(x)$ . Since the tensor category generated by  $\mathcal{W}'_{K(x)}$  is contained in  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ , we have a functorial surjective group morphism

$$\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{W}'_{K(x)}, \eta_{K(x)}) = \{1\}.$$

We conclude that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  acts trivially over  $\mathcal{W}'_{K(x)}$ , and therefore that  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is contained in  $G$ .  $\square$

**Corollary 4.9.** *Theorem 3.1 and Theorem 4.5 are equivalent.*

*Proof.* We have seen in the proof above that Theorem 3.1 implies Theorem 4.5. Corollary 4.8 gives the opposite implication.  $\square$

## 4.1 Finite generic Galois groups

We deduce from Theorem 4.5 the following description of a finite generic Galois group:

**Corollary 4.10.** *The following facts are equivalent:*

1. *There exists a positive integer  $r$  such that the  $q$ -difference module  $\mathcal{M} = (M, \Sigma_q)$  becomes trivial as a  $\tilde{q}$ -difference module over  $K(\tilde{q}, t)$ , with  $\tilde{q}^r = q$ ,  $t^r = x$ .*
2. *There exists a positive integer  $r$  such that for almost all  $v \in \mathcal{C}$  the morphism  $\Sigma_q^{\kappa_v r}$  induces the identity on  $M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ .*
3. *There exists a  $q$ -difference field extension  $\mathcal{F}/K(x)$  of finite degree such that  $\mathcal{M}$  becomes trivial over  $\mathcal{F}$ .*
4. *The (generic) Galois group of  $\mathcal{M}$  is finite.*

*In particular, if  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is finite, it is necessarily cyclic (of order  $r$ , if one chooses  $r$  minimal in the assertions above).*

*Proof.* The equivalence “ $1 \Leftrightarrow 2$ ” follows from Theorem 3.1 applied to the  $\tilde{q}$ -difference module  $(M \otimes K(\tilde{q}, t), \Sigma_q \otimes \sigma_{\tilde{q}})$ , over the field  $K(\tilde{q}, t)$ .

If the generic Galois group is finite, the reduction modulo  $\phi_v$  of  $\Sigma_q^{\kappa_v}$  must be a cyclic operator of order dividing the cardinality of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ . So we have proved that “ $4 \Rightarrow 2$ ”. On the other hand, assertion 2 implies that there exists a basis of  $\mathcal{M}_{K(x)}$  such that the representation of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is given by the group of diagonal matrices, whose diagonal entries are  $r$ -th roots of unity.

Of course, assertion 1 implies assertion 3. The inverse implication follows from the proposition below, applied to a cyclic vector of  $\mathcal{M}_{K(x)}$ .  $\square$

**Lemma 4.11.** *Let  $K$  be a field and  $q$  an element of  $K$  which is not a root of unity. We suppose that there exists a norm  $|\cdot|$  over  $K$  such that  $|q| \neq 1^6$  and we consider a linear  $q$ -difference equations*

$$(4.1) \quad a_\nu(x)y(q^\nu x) + a_{\nu-1}(x)y(q^{\nu-1}x) + \cdots + a_0(x)y(x) = 0$$

*with coefficients in  $K(x)$ . If there exists an algebraic  $q$ -difference extension  $\mathcal{F}$  of  $K(x)$  containing a solution  $f$  of (4.1), then  $f$  is contained in an extension of  $K(x)$  isomorphic to  $K(\tilde{q}, t)$ , with  $\tilde{q}^r = 1$  and  $t^r = x$ .*

*Proof.* Let us look at (4.1) as an equation with coefficients in  $K((x))$ . Then the algebraic solution  $f$  of (4.1) can be identified to a Laurent series in  $\overline{K}((t))$ , where  $\overline{K}$  is the algebraic closure of  $K$  and  $t^r = x$ , for a convenient positive integer  $r$ . Let  $\tilde{q}$  be an element of  $\overline{K}$  such that  $\tilde{q}^r = q$  and that  $\sigma_q(f) = f(\tilde{q}t)$ . We can look at (4.1) as a  $\tilde{q}$ -difference equation with coefficients in  $K(\tilde{q}, t)$ . Then the recurrence relation induced by (4.1) over the coefficients of a formal solution shows that there exist  $f_1, \dots, f_s$  solutions of (4.1) in  $K(\tilde{q})((t))$  such that  $f \in \sum_i \overline{K} f_i$ . It follows that there exists a finite extension  $\tilde{K}$  of  $K(\tilde{q})$  such that  $f \in \tilde{K}((t))$ .

We fix an extension of  $|\cdot|$  to  $\tilde{K}$ , that we still call  $|\cdot|$ . Since  $f$  is algebraic, it is a germ of meromorphic function at 0. Since  $|\tilde{q}| \neq 1$ , the functional equation (4.1) itself allows to show that  $f$  is actually a meromorphic function with infinite radius of meromorphy. Finally,  $f$  can have at worst a pole at  $t = \infty$ , since it is an algebraic function, which actually implies that  $f$  is the Laurent expansion of a rational function in  $K(\tilde{q}, t)$ .  $\square$

## 4.2 Devissage of nonreduced generic Galois groups

Independently of the characteristic of the base field, there is no proper Galois correspondence for generic Galois groups. If  $\mathcal{N} = (N, \Sigma_q)$  is an object of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ , then there exists a normal subgroup  $H$  of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  such that  $H$  acts as the identity on  $N_{K(x)}$  and

$$(4.2) \quad \text{Gal}(\mathcal{N}_{K(x)}, \eta_{K(x)}) \cong \frac{\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})}{H}.$$

In fact, the category  $\langle \mathcal{N}_{K(x)} \rangle^\otimes$  is a full subcategory of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$  and therefore there exists a surjective functorial morphism

$$\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{N}_{K(x)}, \eta_{K(x)}).$$

The kernel of such morphism is the normal subgroup of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  that acts as the identity on  $\mathcal{N}_{K(x)}$ . On the other hand, if  $H$  is a normal subgroup of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ , it is not always possible to find an object  $\mathcal{N}_{K(x)} = (N_{K(x)}, \Sigma_q)$  of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$  such that we have (4.2). This happens because the generic Galois group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  stabilizes all the sub- $q$ -difference modules of the constructions on  $\mathcal{M}_{K(x)}$  but also other submodules, which are not stable by  $\Sigma_q$ . So, if  $H = \text{Stab}(L_{K(x)})$ , for some line  $L_{K(x)}$  in some algebraic construction of  $\mathcal{M}_{K(x)}$ , the orbit of  $L_{K(x)}$  with respect to  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  could be a  $q$ -difference module, allowing to establish (4.2), but in general it won't be.

In spite of the fact that in this setting we do not have a Galois correspondence, we can establish some devissage of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ , when it is not reduced. So let us suppose that the group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is nonreduced, and therefore that the characteristic of  $k$  is  $p > 0$ . Then there exists a maximal reduced subgroup  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  and a short exact sequence of groups:

$$(4.3) \quad 1 \longrightarrow \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \mu_{p^\ell} \longrightarrow 1,$$

for some positive integer  $\ell$ , uniquely determined by the above short exact sequence. We remind that the subgroup  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is normal.

<sup>6</sup>This assumption is always verified if  $K$  is a finite extension of a field of rational functions  $k(q)$ , as in this paper, or if there exists an immersion of  $K$  in  $\mathbb{C}$ .

**Theorem 4.12.** *The subgroup  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  of  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the smallest algebraic subgroup of  $\text{GL}(M_{K(x)})$  whose reduction modulo  $\phi_v$  contains the operators  $\Sigma_q^{\kappa_v p^\ell}$  for almost all  $v \in \mathcal{C}_K$ .*

We first prove two lemmas.

**Lemma 4.13.** *The group  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is contained in the smallest algebraic subgroup  $H$  of  $\text{GL}(M_{K(x)})$  whose reduction modulo  $\phi_v$  contains the operators  $\Sigma_q^{\kappa_v p^\ell}$  for almost all  $v \in \mathcal{C}_K$ .*

*Proof.* Let  $H$  be the smallest algebraic subgroup of  $\text{GL}(M_{K(x)})$  whose reduction modulo  $\phi_v$  contains the operators  $\Sigma_q^{\kappa_v p^\ell}$  for almost all  $v \in \mathcal{C}_K$ . We know that  $H = \text{Stab}(L_{K(x)})$  for some line  $L_{K(x)}$  contained in some object of  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ . Once again, as in the proof of Theorem 4.5, we can find another line  $L'_{K(x)}$ , that defines  $H$  as a stabilizer and which is actually fixed by  $H$ . It follows that  $L'$  generates a  $q$ -difference module  $\mathcal{W}'$  over  $\mathcal{A}$ , that satisfies the hypothesis of Corollary 4.10. We conclude that there exists a nonnegative integer  $\ell' \leq \ell$  such that  $H$  is contained in the kernel of the surjective map:

$$(4.4) \quad \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{W}'_{K(x)}, \eta_{K(x)}) = \mu_{p^{\ell'}},$$

and therefore that  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \subset H$ .  $\square$

**Lemma 4.14.** *Let  $q^{(\ell)} = q^{p^\ell}$ . We consider the  $q^{(\ell)}$ -difference module  $\mathcal{M}_{K(x)}^{(\ell)}$  obtained from  $\mathcal{M}_{K(x)}$  iterating  $\Sigma_q$ , i.e.  $\mathcal{M}_{K(x)}^{(\ell)} = (M_{K(x)}, \Sigma_{q^{(\ell)}})$ , with  $\Sigma_{q^{(\ell)}} = \Sigma_q^{p^\ell}$ . Then  $\text{Gal}(\mathcal{M}_{K(x)}^{(\ell)}, \eta_{K(x)})$  is the smallest algebraic subgroup of  $\text{GL}(M_{K(x)})$  whose reduction modulo  $\phi_v$  contains the operators  $\Sigma_q^{\kappa_v p^\ell}$  for almost all  $v \in \mathcal{C}_K$ .*

*Proof.* Since the characteristic of  $k$  is  $p > 0$ , the order  $\kappa_v$  of  $q_v$  in the residue field  $k_v$  is a divisor of  $p^n - 1$  for some positive integer  $n$ . It follows that the order of  $q^{(\ell)}$  modulo  $v$  is equal to  $\kappa_v$  for almost all  $v \in \mathcal{C}_K$ . Theorem 4.5 allows to conclude, since  $\Sigma_{q^{(\ell)}}^{\kappa_v} = \Sigma_q^{\kappa_v p^\ell}$ .  $\square$

*Proof of Theorem 4.12.* We will prove the statement by induction on  $\ell \geq 0$ , in the short exact sequence (4.3). The statement is trivial for  $\ell = 0$ , since in this case  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ . Let us suppose that  $\ell > 0$  and that the statement is proved for any  $\ell' < \ell$ . In the notation of the lemmas above, we have:

$$\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \subset H.$$

We suppose that the inclusion is strict, otherwise there would be nothing to prove. This means that in (4.4) we have  $\ell' > 0$ :

$$H \hookrightarrow \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \twoheadrightarrow \mu_{p^{\ell'}}.$$

We claim that  $H$  is the smallest subgroup that contains  $\Sigma_q^{\kappa_v p^{\ell'}}$  modulo  $\phi_v$  for almost all  $v$  and therefore that  $H = \text{Gal}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)})$ , because of Lemma 4.14. In fact the smallest subgroup that contains  $\Sigma_q^{\kappa_v p^{\ell'}}$  modulo  $\phi_v$  for almost all  $v$  is contained in  $H$  by definition, while morphism (4.4) proves that  $\Sigma_q^{\kappa_v p^{\ell'}}$  stabilizes the line  $L_{K(x)}$ , considered in Lemma 4.13, modulo  $\phi_v$ . Then Lemma 4.14 implies that  $H = \text{Gal}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)})$ .

Since  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)}) \subset H$ , we have a short exact sequence:

$$1 \longrightarrow \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)}) \longrightarrow \text{Gal}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)}) \longrightarrow \mu_{p^{\ell-\ell'}} \longrightarrow 1.$$

The inductive hypotheses implies that  $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}^{(\ell')}, \eta_{K(x)})$  is the smallest subgroup of  $\text{GL}(M_{K(x)})$  containing the operators  $\Sigma_{q^{(\ell)}}^{\kappa_v p^{\ell-\ell'}} = \Sigma_q^{\kappa_v p^\ell}$ . This ends the proof.  $\square$

We obtain the following corollary:

**Corollary 4.15.** *In the notation of the theorem above:*

- $\text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) = \text{Gal}(\mathcal{M}_{K(x)}^{(\ell)}, \eta_{K(x)})$ .

- Let  $\tilde{K}$  be a finite extension of  $K$  containing a  $p^\ell$ -th root  $q^{1/p^\ell}$  of  $q$ . Then the generic Galois group  $\text{Gal}(\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}, \eta_{\tilde{K}(x^{1/p^\ell})})$  of the  $q^{1/p^\ell}$ -difference module  $\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}$  is reduced and

$$\text{Gal}(\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}, \eta_{\tilde{K}(x^{1/p^\ell})}) \subset \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \tilde{K}(x^{1/p^\ell}).$$

*Proof.* The first statement is a rewriting of Lemma 4.14. We have to prove the second statement. If  $\underline{e}$  is a basis of  $M_{K(x)}$  such that  $\Sigma_q \underline{e} = \underline{e} A(x)$ , then in  $\mathcal{M}_{K(x^{1/p^\ell})} = (M_{K(x^{1/p^\ell})}, \Sigma_{q^{1/p^\ell}} := \Sigma_q \otimes \sigma_{q^{1/p^\ell}})$  we have:

$$\Sigma_{q^{1/p^\ell}}(\underline{e} \otimes 1) = (\underline{e} \otimes 1)A(x).$$

It follows that the generic Galois group  $\text{Gal}(\mathcal{M}_{K(x^{1/p^\ell})}, \eta_{K(x^{1/p^\ell})})$  is the smallest algebraic subgroup of  $\text{GL}(M_{K(x^{1/p^\ell})})$  that contains the operators  $\Sigma_{q^{1/p^\ell}}^{\kappa p^\ell} = \Sigma_q^{\kappa} \otimes 1$ . This proves that

$$\text{Gal}(\mathcal{M}_{\tilde{K}(x^{1/p^\ell})}, \eta_{\tilde{K}(x^{1/p^\ell})}) \subset \text{Gal}_{\text{red}}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \tilde{K}(x^{1/p^\ell}).$$

□

## 5 Grothendieck conjecture on $p$ -curvatures for $q$ -difference modules in characteristic zero, with $q \neq 1$

Let  $K$  be a finitely generated extension of  $\mathbb{Q}$  and  $q \in K \setminus \{0, 1\}$ . The previous results, combined with an improved version of [DV02], give a “curvature” characterization of the generic Galois group of a  $q$ -difference module over  $K(x)$ . We will constantly distinguish three cases:

- $q$  is a root of unity;
- $q$  is transcendental over  $\mathbb{Q}$ ;
- $q$  is algebraic over  $\mathbb{Q}$ , but is not a root of unity.

**Case 1:  $q$  is a root of unity.** If  $q$  is a primitive root of unity of order  $\kappa$ , it is not difficult to prove that:

**Proposition 5.1** ([DV02, Proposition 2.1.2]). *A  $q$ -difference module  $\mathcal{M}_{K(x)}$  over  $K(x)$  is trivial if and only if  $\Sigma_q^\kappa$  is the identity.*

**Case 2:  $q$  is transcendental over  $\mathbb{Q}$ .** If  $q$  is transcendental over  $\mathbb{Q}$ , we can always find an intermediate field  $k$  of  $K/\mathbb{Q}$  such that  $K$  is a finite extension of  $k(q)$ . We are in the situation of Theorem 3.1, that we can rephrase as follows:

**Theorem 5.2.** *A  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  over  $K(x)$  is trivial if and only if there exists a  $k$ -algebra  $\mathcal{A}$  (as in (1.1)) and a  $\Sigma_q$ -stable  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$  such that for almost all cyclotomic places  $v \in \mathcal{C}$  the  $v$ -curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$$

*is the identity.*

**Case 3:  $q$  is algebraic over  $\mathbb{Q}$ , but not a root of unity.** Finally if  $q$  is algebraic, but not a root of unity, we are in the following situation. We call  $Q$  the algebraic closure of  $\mathbb{Q}$  inside  $K$ ,  $\mathcal{O}_Q$  the ring of integer of  $Q$ ,  $v$  a finite places of  $Q$  and  $\pi_v$  a  $v$ -adic uniformizer. For almost all  $v$ , the order  $\kappa_v$  of  $q$  modulo  $v$ , as a root of unity, and the positive integer power  $\phi_v$  of  $\pi_v$ , such that  $\phi_v^{-1}(1 - q^{\kappa_v})$  is a unit of  $\mathcal{O}_Q$ , are well defined. The field  $K$  has the form  $Q(\underline{a}, b)$ , where  $\underline{a} = (a_1, \dots, a_r)$  is a transcendent basis of  $K/Q$  and  $b$  is a primitive element of the algebraic extension  $K/Q(\underline{a})$ . Choosing conveniently the set of generators  $\underline{a}, b$ , we can always find an algebra  $\mathcal{A}$  of the form:

$$(5.1) \quad \mathcal{A} = \mathcal{O}_Q \left[ \underline{a}, b, x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right],$$

for some  $P(x) \in \mathcal{O}_Q[\underline{a}, b, x]$ , and a  $\Sigma_q$ -stable  $\mathcal{A}$ -lattice  $M$  of  $\mathcal{M}_{K(x)}$ , so that we can consider the linear operator

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v),$$

that we will call the  $v$ -curvature of  $\mathcal{M}_{K(x)}$ -modulo  $\phi_v$ . Notice that  $\mathcal{O}_Q/(\phi_v)$  is not an integral domain in general.

We are going to prove the following:

**Theorem 5.3.** *A  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  over  $K(x)$  is trivial if and only if there exists a  $k$ -algebra  $\mathcal{A}$  as above and a  $\Sigma_q$ -stable  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$  such that for almost all finite places  $v$  of  $Q$  the  $v$ -curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$$

*is the identity.*

The theorem above is proved in [DV02] under the assumption that  $K$  is a number field, *i.e.* that  $Q = K$ . Here  $K$  is only a finitely generated extension of  $\mathbb{Q}$ . The proof in this more general case is given in §5.1 below. Notice that the proofs in [DV02] require  $K$  to be a number field, so that the proofs below relies crucially on [DV02], but is not a generalization of the arguments in *loc. cit.*

**A unified statement.** In order to give a unified statement for the three theorems above we introduce the following notation:

- if  $q$  is a root of unity of order  $\kappa$ , we can take  $\mathcal{C}$  to be the set containing only the trivial valuation  $v$  on  $K$ ,  $\mathcal{A}$  to be a  $\sigma_q$ -stable extension of  $K[x]$  obtained inverting a convenient polynomial,  $(\phi_v) = (0)$  and  $\kappa_v = \kappa$ ;
- if  $q$  is transcendental,  $\mathcal{C}$  is the set of cyclotomic places as in the notation of the previous sections;
- if  $q$  is algebraic, not a root of unity, we set  $\mathcal{C}$  to be the set of finite places of  $Q$ .

Therefore we have:

**Theorem 5.4.** *A  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  over  $K(x)$  is trivial if and only if there exists a  $k$ -algebra  $\mathcal{A}$  as above and a  $\Sigma_q$ -stable  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$  such that for any  $v$  in a cofinite nonempty subset of  $\mathcal{C}$ , the  $v$ -curvature*

$$\Sigma_q^{\kappa_v} : M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v) \longrightarrow M \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$$

*is the identity.*

## 5.1 Proof of Theorem 5.3

We only need to prove Theorem 5.3 under the assumption that  $K$  is not a number field. The proof (*cf.* the two subsections below) will repose on [DV02, Theorem 7.1.1], which is exactly the same statement plus the extra assumption that  $K$  is a number field.

**Global nilpotence.** In this and in the following subsection we assume that:

( $\mathcal{H}$ )  $K$  is a transcendental finitely generated extension of  $\mathbb{Q}$  and  $q$  is an algebraic number.

**Proposition 5.5.** *Under the hypothesis ( $\mathcal{H}$ ), for a  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  we have:*

1. *If  $\Sigma_q^{\kappa_v}$  induces a unipotent linear morphism on  $M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v)$  for infinitely many finite places  $v$  of  $Q$ , then the  $q$ -difference module  $\mathcal{M}_{K(x)}$  is regular singular.*
2. *If there exists a set of finite places  $v$  of  $Q$  of Dirichlet density 1 such that  $\Sigma_q^{\kappa_v}$  induces a unipotent linear morphism on  $M \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v)$ , then the  $q$ -difference module  $\mathcal{M}_{K(x)}$  is regular singular and its exponents at 0 and  $\infty$  are in  $q^{\mathbb{Z}}$ .*
3. *If  $\Sigma_q^{\kappa_v}$  induces the identity on  $M_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}/(\pi_v)$  for almost all finite places  $v$  of  $Q$  in a set of Dirichlet density 1, then the  $q$ -difference modules  $\mathcal{M}_{K((x))}$  and  $\mathcal{M}_{K((1/x))}$  are trivial.*

We recall that a subset  $S$  of the set of finite places  $\mathcal{C}$  of  $Q$  has Dirichlet density 1 if

$$(5.2) \quad \limsup_{s \rightarrow 1^+} \frac{\sum_{v \in S, v|p} p^{-sf_v}}{\sum_{v \in S_f, v|p} p^{-sf_v}} = 1,$$

where  $f_v$  is the degree of the residue field of  $v$  over  $\mathbb{F}_p$ .

*Proof.* The proof is the same as [DV02, Theorem 6.2.2 and Proposition 6.2.3] (cf. also Theorem 2.3 and Corollary 3.6 above). The idea is that one has to choose a basis  $\underline{e}$  of  $M_{\mathcal{A}}$  such that  $\Sigma_q \underline{e} = \underline{e}A(x)$  for some  $A(x) \in \mathrm{GL}_{\nu}(\mathcal{A})$ . Then the hypothesis on the reduction of  $\Sigma_q^{\kappa_v}$  modulo  $\pi_v$  forces  $A(x)$  not to have poles at 0 and  $\infty$ . Moreover we deduce that  $A(0), A(\infty) \in \mathrm{GL}_{\nu}(K)$  are actually semisimple matrices, whose eigenvalues are in  $q^{\mathbb{Z}}$ .  $\square$

**End of the proof of Theorem 5.3.** We assume  $(\mathcal{H})$ . We will deduce Theorem 5.3 from the analogous results in [DV02], where  $K$  is assumed to be a number field. To do so, we will consider the transcendence basis of  $K/Q$  as a set of parameter that we will specialize in the algebraic closure of  $Q$ . We will need the following (very easy) lemma:

**Lemma 5.6.** *Let  $F$  be a field and  $q$  be an element of  $F$ , not a root of unity. We consider a  $q$ -difference system  $Y(qx) = A_0(x)Y(x)$  such that  $A_0(x) \in \mathrm{GL}_{\nu}(F(x))$ , zero is not a pole of  $A_0(x)$  and such that  $A_0(0)$  is the identity matrix. Then, for any norm  $|\cdot|$  (archimedean or ultrametric) over  $F$  such that  $|q| > 1$  the formal solution*

$$Z_0(x) = \left( A_0(q^{-1}x)A_0(q^{-2}x)A_0(q^{-3}x) \dots \right)$$

*of  $Y(qx) = A_0(x)Y(x)$  is a germ of an analytic fundamental solution at zero having infinite radius of meromorphy.*<sup>7</sup>

*Proof.* Since  $|q| > 1$  the infinite product defining  $Z_0(x)$  is convergent in the neighborhood of zero. The fact that  $Z_0(x)$  is a meromorphic function with infinite radius of meromorphy follows from the functional equation  $Y(qx) = A_0(x)Y(x)$  itself.  $\square$

*Proof of Theorem 5.3.* One side of the implication in Theorem 5.3 is trivial. So we suppose that  $\Sigma_q^{\kappa_v}$  induces the identity on  $M_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$  for almost all finite places  $v$  of  $Q$ , and we prove that  $\mathcal{M}_{\mathcal{A}}$  becomes trivial over  $K(x)$ . The proof is divided into steps:

*Step 0. Reduction to a purely transcendental extension  $K/Q$ .* Let  $\underline{a}$  be a transcendence basis of  $K/Q$  and  $b$  is a primitive element of  $K/Q(\underline{a})$ , so that  $K = Q(\underline{a}, b)$ . The  $q$ -difference field  $K(x)$  can be considered as a trivial  $q$ -difference module over the field  $Q(\underline{a})(x)$ . By restriction of scalars, the module  $\mathcal{M}_{K(x)}$  is also a  $q$ -difference module over  $Q(\underline{a})(x)$ . Since the field  $K(x)$  is a trivial  $q$ -difference module over  $Q(\underline{a})(x)$ , we have:

- the module  $\mathcal{M}_{K(x)}$  is trivial over  $K(x)$  if and only if it is trivial over  $Q(\underline{a})(x)$ ;
- under the present hypothesis, there exist an algebra  $\mathcal{A}'$  of the form

$$(5.3) \quad \mathcal{A}' = \mathcal{O}_Q \left[ \underline{a}, x, \frac{1}{R(x)}, \frac{1}{R(qx)}, \dots \right], \quad R(x) \in \mathcal{O}_Q[\underline{a}, x],$$

and a  $\mathcal{A}'$ -lattice  $\mathcal{M}_{\mathcal{A}'}$  of  $q$ -difference module  $\mathcal{M}_{K(x)}$  over  $Q(\underline{a})(x)$ , such that  $\mathcal{M}_{\mathcal{A}'} \otimes_{\mathcal{A}'} Q(\underline{a}, x) = \mathcal{M}_{K(x)}$  as a  $q$ -difference module over  $Q(\underline{a}, x)$  and  $\Sigma_q^{\kappa_v}$  induces the identity on  $\mathcal{M}_{\mathcal{A}'} \otimes_{\mathcal{A}'} \mathcal{A}/(\phi_v)$ , for almost all places  $v$  of  $Q$ .

For this reason, we can actually assume that  $K$  is a purely transcendental extension of  $Q$  of degree  $d > 0$  and that  $\mathcal{A} = \mathcal{A}'$ . We fix an immersion of  $Q \hookrightarrow \overline{\mathbb{Q}}$ , so that we will think to the transcendental basis  $\underline{a}$  as a set of parameter generically varying in  $\overline{\mathbb{Q}}^d$ .  $\square$

<sup>7</sup>In the sense that the entries of  $Z_0(x)$  are quotient of two entire analytic functions with respect to  $|\cdot|$ .



*Step 0bis. Initial data.* Let  $K = Q(\underline{a})$  and  $q$  be a nonzero element of  $Q$ , which is not a root of unity. We are given a  $q$ -difference module  $\mathcal{M}_{\mathcal{A}}$  over a convenient algebra  $\mathcal{A}$  as above, such that  $K(x)$  is the field of fraction of  $\mathcal{A}$  and such that  $\Sigma_q^{\kappa_v}$  induces the identity on  $M_{\mathcal{A}} \otimes \mathcal{O}_q/(\phi_v)$ , for almost all finite places  $v$ . We fix a basis  $\underline{e}$  of  $\mathcal{M}_{\mathcal{A}}$ , such that  $\Sigma_q \underline{e} = \underline{e} A^{-1}(x)$ , with  $A(x) \in \mathrm{GL}_{\nu}(\mathcal{A})$ . We will rather work with the associated  $q$ -difference system:

$$(5.4) \quad Y(qx) = A(x)Y(x).$$

It follows from Proposition 5.5 that  $\mathcal{M}_{K(x)}$  is regular singular, with no logarithmic singularities, and that its exponents are in  $q^{\mathbb{Z}}$ . Enlarging a little bit the algebra  $\mathcal{A}$  (more precisely replacing the polynomial  $R$  by a multiple of  $R$ ), we can suppose that both 0 and  $\infty$  are not poles of  $A(x)$  and that  $A(0), A(\infty)$  are diagonal matrices with eigenvalues in  $q^{\mathbb{Z}}$  (cf. [Sau00, §2.1]).  $\square$

*Step 1. Construction of canonical solutions at 0.* We construct a fundamental matrix of solutions, applying the Frobenius algorithm to this particular situation (cf. [vdPS97] or [Sau00, §1.1]). There exists a shearing transformation  $S_0(x) \in \mathrm{GL}_{\nu}(K[x, x^{-1}])$  such that

$$S_0^{-1}(qx)A(x)S_0(x) = A_0(x)$$

and  $A_0(0)$  is the identity matrix. In particular, the matrix  $S_0(x)$  can be written as a product of invertible constant matrices and diagonal matrix with integral powers of  $x$  on the diagonal. Once again, up to a finitely generated extension of the algebra  $\mathcal{A}$ , obtained inverting a convenient polynomial, we can suppose that  $S_0(x) \in \mathrm{GL}_{\nu}(\mathcal{A})$ .

Notice that, since  $q$  is not a root of unity, there always exists a norm, non necessarily archimedean, on  $Q$  such that  $|q| > 1$ . We can always extend such a norm to  $K$ . Then the system

$$(5.5) \quad Z(qx) = A_0(x)Z(x)$$

has a unique convergent solution  $Z_0(x)$ , as in Lemma 5.6. This implies that  $Z_0(x)$  is a germ of a meromorphic function with infinite radius of meromorphy. So we have the following meromorphic solution of  $Y(qx) = A(x)Y(x)$ :

$$Y_0(x) = \left( A_0(q^{-1}x)A_0(q^{-2}x)A_0(q^{-3}x) \dots \right) S_0(x).$$

We remind that this formal infinite product represent a meromorphic fundamental solution of  $Y(qx) = A(x)Y(x)$  for any norm over  $K$  such that  $|q| > 1$  (cf. Lemma 5.6).  $\square$

*Step 2. Construction of canonical solutions at  $\infty$ .* In exactly the same way we can construct a solution at  $\infty$  of the form  $Y_{\infty}(x) = Z_{\infty}(x)S_{\infty}(x)$ , where the matrix  $S_{\infty}$  belongs to  $GL_{\nu}(K[x, x^{-1}]) \cap \mathrm{GL}_{\nu}(\mathcal{A})$  and has the same form as  $S_0(x)$ , and  $Z_{\infty}(x)$  is analytic in a neighborhood of  $\infty$ , with  $Z_{\infty}(\infty) = 1$ :

$$Y_{\infty}(x) = \left( A_{\infty}(x)A_{\infty}(qx)A_{\infty}(q^2x) \dots \right) S_{\infty}(x).$$

$\square$

*Step 3. The Birkhoff matrix.* To summarize we have constructed two fundamental matrices of solutions,  $Y_0(x)$  at zero and  $Y_{\infty}(x)$  at  $\infty$ , which are meromorphic over  $\mathbb{A}_K^1 \setminus \{0\}$  for any norm on  $K$  such that  $|q| > 1$ , and such that their set of poles and zeros is contained in the  $q$ -orbits of the set of poles at zeros of  $A(x)$ . The Birkhoff matrix

$$B(x) = Y_0^{-1}(x)Y_{\infty}(x) = S_0(x)^{-1}Z_0(x)^{-1}Z_{\infty}(x)S_{\infty}(x)$$

is a meromorphic matrix on  $\mathbb{A}_K^1 \setminus \{0\}$  with elliptic entries:  $B(qx) = B(x)$ . All the zeros and poles of  $B(x)$ , other than 0 and  $\infty$ , are contained in the  $q$ -orbit of zeros and poles of the matrices  $A(x)$  and  $A(x)^{-1}$ .  $\square$

*Step 4. Rationality of the Birkhoff matrix.* Let us choose  $\underline{a} = (\alpha_1, \dots, \alpha_r)$ , with  $\alpha_i$  in the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ , so that we can specialize  $\underline{a}$  to  $\underline{a}$  in the coefficients of  $A(x), A(x)^{-1}, S_0(x), S_{\infty}(x)$  and that the specialized matrices are still invertible. Then we obtain a  $q$ -difference system with coefficients in  $Q(\underline{a})$ . It follows from Lemma 5.6 that for any norm on  $Q(\underline{a})$  such that  $|q| > 1$  we can specialize  $Y_0(x), Y_{\infty}(x)$  and therefore  $B(x)$  to matrices with meromorphic entries on  $Q(\underline{a})^*$ . We will write  $A^{(\underline{a})}(x), Y_0^{(\underline{a})}(x)$ , etc. for the specialized matrices.

Since  $A_{\kappa_v}(x)$  is the identity modulo  $\phi_v$ , the same holds for  $A_{\kappa_v}^{(\underline{\alpha})}(x)$ . Therefore the reduced system has zero  $\kappa_v$ -curvature modulo  $\phi_v$  for almost all  $v$ . We know from [DV02], that  $Y_0^{(\underline{\alpha})}(x)$  and  $Y_\infty^{(\underline{\alpha})}(x)$  are the germs at zero of rational functions, and therefore that  $B^{(\underline{\alpha})}(x)$  is a constant matrix in  $\mathrm{GL}_\nu(Q(\underline{\alpha}))$ .

As we have already pointed out,  $B(x)$  is  $q$ -invariant meromorphic matrix on  $\mathbb{P}_K^1 \setminus \{0, \infty\}$ . The set of its poles and zeros is the union of a finite numbers of  $q$ -orbits of the forms  $\beta q^\mathbb{Z}$ , such that  $\beta$  is algebraic over  $K$  and is a pole or a zero of  $A(x)$  or  $A(x)^{-1}$ . If  $\beta$  is a pole or a zero of an entry  $b(x)$  of  $B(x)$  and  $h_\beta(x), k_\beta(x) \in Q[\underline{a}, x]$  are the minimal polynomials of  $\beta$  and  $\beta^{-1}$  over  $K$ , respectively, then we have:

$$b(x) = \lambda \frac{\prod_\gamma \prod_{n \geq 0} h_\gamma(q^{-n}x) \prod_{n \geq 0} k_\gamma(1/q^n x)}{\prod_\delta \prod_{n \geq 0} h_\delta(q^{-n}x) \prod_{n \geq 0} k_\delta(1/q^n x)},$$

where  $\lambda \in K$  and  $\gamma$  and  $\delta$  vary in a system of representatives of the  $q$ -orbits of the zeroes and the poles of  $b(x)$ , respectively. We have proved that there exists a Zariski open set of  $\overline{\mathbb{Q}}^d$  such that the specialization of  $b(x)$  at any point of this set is constant. Since the factorization written above must specialize to a convergent factorization of the same form of the corresponding element of  $B^{(\underline{\alpha})}(x)$ , we conclude that  $b(x)$ , and therefore  $B(x)$  is a constant.  $\square$

The fact that  $B(x) \in \mathrm{GL}(K)$  implies that the solutions  $Y_0(x)$  and  $Y_\infty(x)$  glue to a meromorphic solution on  $\mathbb{P}_K^1$  and ends the proof of Theorem 5.3.  $\square$

## 5.2 Generic Galois group

For any field  $K$  of zero characteristic, any  $q \in K \setminus \{0, 1\}$  and any  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  we can define as in the previous sections the generic Galois group  $\mathrm{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ . If  $K$  is a finitely generated extension of  $\mathbb{Q}$ , in the notation of Theorem 5.4, we have:

**Theorem 5.7.** *The generic Galois group  $\mathrm{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is the smallest algebraic subgroup of  $\mathrm{GL}(M_{K(x)})$  that contains the  $v$ -curvatures of the  $q$ -difference module  $\mathcal{M}_{K(x)}$  modulo  $\phi_v$ , for all  $v$  in a nonempty cofinite subset of  $\mathcal{C}$ .*

The group  $\mathrm{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$  is a stabilizer of a line  $L_{K(x)}$  in a construction  $\mathcal{W}_{K(x)} = (W_{K(x)}, \Sigma_q)$  of  $\mathcal{M}_{K(x)}$ . The statement above says that we can find a  $\sigma_q$ -stable algebra  $\mathcal{A} \subset K(x)$  of one of the forms described above, and a  $\Sigma_q$ -stable  $\mathcal{A}$ -lattice  $M$  of  $M_{K(x)}$  such that  $M$  induces an  $\mathcal{A}$ -lattice  $L$  of  $L_{K(x)}$  and  $W$  of  $W_{K(x)}$  with the following properties: the reduction modulo  $\phi_v$  of  $\Sigma_q^{\kappa_v}$  stabilizes  $L \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$  inside  $W \otimes_{\mathcal{A}} \mathcal{A}/(\phi_v)$ , for any  $v$  in a nonempty cofinite subset of  $\mathcal{C}$ .

Theorem 5.7 has been proved in [Hen96, Chapter 6] when  $q$  is a root of unity, in the previous sections when  $q$  is transcendental and in [DV02] when  $q$  is algebraic and  $K$  is a number field. The remaining case (*i.e.*  $q$  algebraic and  $K$  is transcendental finitely generated over  $\mathbb{Q}$ ) is proved exactly as Theorem 4.5 and [DV02, Theorem 10.2.1].

## 5.3 Generic Galois group of a $q$ -difference module over $\mathbb{C}(x)$ , for $q \neq 0, 1$

We deduce from the previous section a curvature characterization of the generic Galois group of a  $q$ -difference module over  $\mathbb{C}(x)$ , for  $q \in \mathbb{C} \setminus \{0, 1\}$ .<sup>8</sup>

Let  $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$  be a  $q$ -difference module over  $\mathbb{C}(x)$ . We can consider a finitely generated extension of  $K$  of  $\mathbb{Q}$  such that there exists a  $q$ -difference module  $\mathcal{M}_{K(x)} = (M_{K(x)}, \Sigma_q)$  satisfying  $\mathcal{M}_{\mathbb{C}(x)} = \mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x)$ . First of all let us notice that:

**Lemma 5.8.** *The  $q$ -difference module  $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$  is trivial if and only if  $\mathcal{M}_{K(x)}$  is trivial.*

*Proof.* If  $\mathcal{M}_{K(x)}$  is trivial, then  $\mathcal{M}_{\mathbb{C}(x)}$  is of course trivial. The inverse statement is equivalent to the following claim. If a linear  $q$ -difference system  $Y(qx) = A(x)Y(x)$ , with  $A(x) \in \mathrm{GL}_\nu(K(x))$ , has a fundamental solution  $Y(x) \in \mathrm{GL}_\nu(\mathbb{C}(x))$ , then  $Y(x)$  is actually defined over  $K$ . In fact, the system  $Y(qx) = A(x)Y(x)$  must be regular singular with exponents in  $q^\mathbb{Z}$ , therefore the Frobenius algorithm allows to construct a solution  $\tilde{Y}(x) \in \mathrm{GL}_\nu(K((x)))$ . We can look at  $\tilde{Y}(x)$  as an element of  $\mathrm{GL}_\nu(\mathbb{C}((x)))$ . Then there must exist a constant matrix  $C \in \mathrm{GL}_\nu(\mathbb{C})$  such that  $Y(x) = C\tilde{Y}(x)$ . This proves that  $\tilde{Y}(x)$  is the expansion of a matrix with entries in  $K(x)$ .  $\square$

<sup>8</sup>All the statements in this subsection remain true if one replace  $\mathbb{C}$  with any field of characteristic zero.

With an abuse of language, Theorem 5.4 can be rephrased as:

**Theorem 5.9.** *The  $q$ -difference module  $\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q)$  is trivial if and only if there exists a nonempty cofinite set of curvatures of  $\mathcal{M}_{K(x)}$ , that are all zero.*

We can of course define as in the previous sections a generic Galois group  $\text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ . A noetherianity argument, that we have already used several times, shows the following:

**Proposition 5.10.** *In the notation above we have:*

$$\text{Gal}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) \subset \text{Gal}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x).$$

Moreover there exists a finitely generated extension  $K'$  (resp.  $K''$ ) of  $K$  such that

$$\text{Gal}(\mathcal{M}_{K(x)} \otimes_{K(x)} K'(x), \eta_{K'(x)}) \otimes_{K'(x)} \mathbb{C}(x) \cong \text{Gal}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}).$$

Choosing  $K$  large enough, we can assume that  $K = K'$ , which we will do implicitly in the following informal statement. We can deduce from Theorem 5.7:

**Theorem 5.11.** *The generic Galois group  $\text{Gal}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$  is the smallest algebraic subgroup of  $\text{GL}(M_{\mathbb{C}(x)})$  that contains a nonempty cofinite set of curvatures of the  $q$ -difference module  $\mathcal{M}_{K(x)}$ .*

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